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ON ESTIMATING THE SOLUTIONS OF HYPOELLIPTIC DIFFERENTIAL EQUATIONS NEAR THE PLANE BOUNDARY[1]

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Introduction.

This paper arose from an attempt to extend the results of Hörmander [5] for hypoelliptic differential operators with constant coefficients to hypoelliptic differential operators with variable coefficients[2]. The results are, however, mainly of negative character in the sense that we disprove an inequality that holds in the elliptic case and that one would have reason to believe to hold also in the hypoelliptic case; and they give thus by no means a complete picture of the situation. Also we consider mostly Dirichlet boundary conditions only, and—what is worse—in the neighborhood of a plane portion of the boundary.

Let R^n be the n-dimensional real Cartesian space; denote elements of R^n by $x=(x_1,\ldots,x_n)=(x',t)$ with $x'=(x_1,\ldots,x_{n-1})$ and $t=x_n$, and elements of the dual space by $\xi=(\xi_1,\ldots,\xi_n)=(\xi',\tau)$ with $\xi'=(\xi_1,\ldots,\xi_{n-1})$ and $\tau=\xi_n$. Let $A=A(D)=A(D_{x'},D_t)$ be a properly hypoelliptic differential operator with constant coefficients in R^n. Following Schechter [11] (cf. [1], [5]), the appelation "properly" means that the "root condition" holds: The number of roots $\tau_r(\xi')$ of the equation $A(\xi',\tau)=0$ with positive (negative) imaginary parts is independent of ξ' for ξ' outside some compact set. Denote this number by m_+ (m_-) and put further $m=m_+ + m_-$ which is the normal order of A. We may suppose that $\operatorname{Im}\tau_r(\xi')>0$ for $r=1,\ldots,m_+$ and <0 for $r=m_+ +1,\ldots,m$. Consider now C^∞ functions u defined in the closed halfspace $\overline{R^n_+}=\{x \mid t\geqq 0\}$ and

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[2] By hypoelliptic we always mean formally hypoelliptic in the sense of [4], [7]: a differential operator $A=A(x,D)$ is said to be hypoelliptic if

1° $\forall x\, \forall y\; A(x,\xi)\sim A(y,\xi),\ \xi\to\infty$,
2° $\forall x\, \forall \alpha\neq 0\; A^{\alpha}(x,\xi)=o\big(A(x,\xi)\big),\ \xi\to\infty$;

A is said to be elliptic if moreover $\forall x\; A(x,\xi)\sim|\xi|^m,\ \xi\to\infty$ for some m (= the order of A)

with compact support. If A is elliptic, then $m_+ = m_-$, and the following well-known inequality:

$$(1) \qquad |||u, R^n{}_+||| \leqq C(||Au, R^n{}_+|| + ||u, R^n{}_+||)$$

holds with a suitable constant C for all u satisfying the condition

$$(2) \qquad D_t^{j-1} u = 0, \qquad j = 1, \ldots, m_+ ,$$

on the plane $R^{n-1} = \{x \mid t = 0\}$. (For more general results see e.g. [1].) Here we have put

$$(3) \qquad ||u, X|| = \left(\int_X |u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

and

$$(4) \qquad |||u, X||| = \sum_{|\alpha| \leqq m} \left(\int_X |D_\alpha u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

for any open set X of R^n. The question is now how to extend this result to the hypoelliptic case. We observe that in the elliptic case the norm $|||u, X|||$ is equivalent to the norm

$$\operatorname{glb} (||Au_1, R^n|| + ||u_1, R^n||)$$

where the glb is taken over all functions u_1 in R^n whose restrictions to X equal u. So it seems quite reasonable to take this as a definition of $|||u, X|||$ in the non-elliptic case:

$$(5) \qquad |||u, X||| = \operatorname{glb} (||Au_1, R^n|| + ||u_1, R^n||) .$$

If one would then be able to prove (1), with the norm defined as by (5), for all u satisfying (2), one would, exactly as in the elliptic case, be able to extend (1) to hypoelliptic operators with variable coefficients, and as a consequence obtain the corresponding regularity theorem (see Section 4). The main contribution of this paper is now that (1), with the norm given as by (5), is not true in general. This phenomenon, in a somewhat different situation, was however already noted by Thomée [12] who considers homogeneous operators and takes $n = 2$; he also admits constants C that depend on the diameter of the support of u. But we prove also that, more precisely, (1) holds if and only if the following condition is fulfilled: Let $B(\xi)$ be any function that is a polynomial of degree $< m_+$ in τ. Expand $B(\xi)/A(\xi)$ in partial fractions:

$$(6) \qquad B(\xi)/A(\xi) = B_+(\xi)/A_+(\xi) + B_-(\xi)/A_-(\xi) ,$$

where

$$A_+(\xi) = \prod_{r=1}^{m_+} (\tau - \tau_r(\xi'))$$

and

$$A_-(\xi) = \prod_{r=m_++1}^{m} (\tau - \tau_r(\xi')) .$$

Then we have the following inequality:

$$\int |B_-(\xi)/A_-(\xi)|^2 \, d\tau \leqq C_*^2 \int |B_+(\xi)/A_+(\xi)|^2 \, d\tau \tag{7}$$

for all ξ' outside some compact set with some constant C_* (independent of B and ξ').

Let us consider some special cases. Suppose $m_+ = m_-$. Then we may take $B_+(\xi) = B_-(\xi) = 1$, and we hence obtain the necessary condition

$$\int 1/|A_-(\xi)|^2 \, d\tau \leqq C_*^2 \int 1/|A_+(\xi)|^2 \, d\tau .$$

If $m_+ = m_- = 1$, this condition reads

$$|\mathrm{Im}\, \tau_1(\xi')| \leqq C_*^2 |\mathrm{Im}\, \tau_2(\xi')|$$

and is in fact also sufficient. It follows that (1) cannot hold for the special operator

$$A = (D_t - i\Delta^2)(D_t + i\Delta), \qquad \Delta = D_1^2 + \ldots + D_{n-1}^2 , \tag{8}$$

which is obviously properly hypoelliptic. Finally, (7) is fulfilled whenever A is elliptic. This can be seen as follows. Suppose A is homogeneous. (This is no restriction.) Then it is obviously sufficient to establish (7) for $|\xi'| = 1$. But (7) holds trivially for every $\xi' \neq 0$, with a constant C_* depending on ξ'. It is now easily verified that C_* is a continuous function of ξ', and hence it will be bounded for $|\xi'| = 1$. This proves the assertion.

Next one would perhaps suspect that the definition of the norm given above by (5) is not at all the "right" one. The following (weaker) norm would perhaps be more natural and more like the one in the original definition for the elliptic case:

$$|||u, X|||' = \sum_{k=1}^{p} \|A_k u, X\| , \tag{9}$$

where $\{A_k\}_{k=1}^p$ is any basis of the vector space of differential operator weaker than A (in the sense of [3]). In the elliptic case we may take $\{A_k\}_{k=1}^p = \{D_\alpha\}_{|\alpha| \leqq m}$ and (9) goes over into (4). We do not know if the two norms $|||u, X|||'$ and $|||u, X|||$ are equivalent in general. We show now that (1) can not hold in general even with $|||u,X|||$ replaced by $|||u,X|||'$. Take A to be the operator (8). Then

$$A_1 = (D_t+i\Delta^2)(D_t+i\Delta)$$

is weaker than A, and $|||u,R^n{}_+|||$ is equivalent to the norm

$$\|A_1u,R^n{}_+\|+\|u,R^n{}_+\| .$$

Hence if

$$(1') \qquad |||u,R^n{}_+|||' \leqq C(\|Au,R^n{}_+\|+\|u,R^n{}_+\|)$$

would hold true then also (1) would hold true, since clearly

$$\|A_1u,R^n{}_+\| \leqq D|||u,R^n{}_+|||' ,$$

which gives the contradiction.

Up to now we have assumed that the constant C in (1) is independent of the diameter of the support of u. But even if we admit, as in the case considered by Thomée [12], a constant C depending on the diameter, (1) or (1′) will still not be true in general. This we will show in the case of the special operator (8) in Section 3.

In Section 1 we reduce the problem to the case $n=1$. This case is then studied in Section 2 where the equivalent of (7) is established for ordinary differential operators or more precisely families of such operators. In Section 3 we study the special operator (8). In Section 4 we prove the regularity theorem for operators for which the condition (7) is fulfilled. (It is possible that the regularity theorem is true also without this assumption.) Finally, in Section 5, we indicate briefly how to extend the results of the paper in several directions.

I would like to thank Prof. Hörmander for valuable advice in connection with the preparation of this paper.

1. Reduction to the case $n=1$.

Let $A=A(D)=A(D_{x'},D_t)$ be any differential operator with constant coefficients, satisfying the root condition (see Introduction). (It is also assumed that the coefficient of D_t^m is a constant.) Taking the Fourier transform with respect to ξ', we get a family of ordinary differential operators $A(\xi',D_t)$. Denote by $|||v,X|||_{\xi'}$ the corresponding norm:

$$|||v,X|||_{\xi'} = \operatorname{glb} |||v_1,R|||_{\xi'}, \qquad v_1 = v \text{ in } X ,$$

where

$$|||v_1,R|||_{\xi'} = \|v_1,R\|+\|A(\xi',D_t)v_1,R\| ,$$

$\|v,X\|$ being defined by (3). We note the following important formula:

$$|||u, R^n{}_+||| \sim \left(\int |||\hat{u}(\xi', .), R_+|||_{\xi'}{}^2 \, d\xi'\right)^{\frac{1}{2}}, \tag{10}$$

the equivalence being uniform in ξ' (for the proof cf. [9, formula (10)]).

LEMMA 1. *The inequality* (1) *holds for all u satisfying* (2) *if and only if for every ξ' with some constant C independent of ξ' we have*

$$|||v, R_+|||_{\xi'} \leqq C(\|A(\xi', D_t)v, R_+\| + \|v, R_+\|) \tag{11}$$

for all C^∞ functions v on R_+ with compact support satisfying the condition

$$D_t^{j-1}v(0) = 0, \quad j = 1, \ldots, m_+ . \tag{12}$$

PROOF. i) Suppose (1) is valid. Let φ be an arbitrary C^∞ function on R^{n-1} with compact support. Apply (1) to the function $u(x) = \varphi(x')v(t)$. Taking Fourier transforms with respect to ξ', we obtain, formula (10) in view,

$$\int |||v, R_+|||_{\xi'}{}^2 \, |\hat{\varphi}(\xi')|^2 \, d\xi' \leqq C^2 \left(\int (\|A(\xi', D_t)v, R_+\|^2 + \|v, R_+\|^2) \, |\hat{\varphi}(\xi')|^2 \, d\xi'\right),$$

from which (11) easily follows.

ii) Suppose (11) is valid. Apply (11) to the function $v(t) = \hat{u}(\xi', t)$ and integrate the square with respect to ξ'. In view of (10), (1) immediately follows.

LEMMA 2. *Let S be a compact set in R^{n-1}. There exists a constant C such that for every $\xi' \in S$ we have*

$$|||v, R_+|||_{\xi'} \leqq C(\|A(\xi', D_t)v, R_+\| + \|v, R_+\|) \tag{13}$$

for all v (no boundary conditions whatsoever!).

PROOF. It is clear that (13) holds for every ξ', with a constant C in general depending on ξ'. But C is obviously a continuous function of ξ' so it will be bounded in S. This proves the assertion.

COROLLARY. *In the formulation of Lemma* 1 *it is sufficient to require that* (11) *holds outside some compact set.*

Next we assume that the following condition is also fulfilled:

$$\varliminf_{\xi' \to \infty} |\operatorname{Im} \tau_r(\xi')| > 0 , \tag{14}$$

which is certainly true if A is hypoelliptic.

LEMMA 3. *Suppose that* (14) *is fulfilled. The inequality* (1) *holds for all u satisfying* (2) *if and only if for all ξ' outside some compact set we have*

$$|||v, R_+|||_{\xi'} \leqq C\|A(\xi', D_t)v, R_+\|$$

for all v *satisfying* (12).

In view of the corollary of Lemma 2, it is clear that Lemma 3 will follow immediately from the corollary of the following lemma.

LEMMA 4. *Let* $A = A(D_t)$ *be any ordinary differential operator with constant coefficients such that* $A(\tau) \neq 0$ *for* τ *real. Let* m_+ (m_-) *be the number of roots of the equation* $A(\tau) = 0$ *with positive (negative) imaginary parts; put*

$$A_+(\tau) = \prod_{r=1}^{m_+} (\tau - \tau_r)\big(A_-(\tau)\big) = \prod_{r=m_++1}^{m} (\tau - \tau_r) . \tag{15}$$

Then we have

$$\|v, R_+\|_{A_+} \leqq \|Av, R_+\|_{A_-^{-1}} \tag{16}$$

for all v *satisfying* (12).

COROLLARY. *The inequality*

$$\|v, R_+\| \leqq \big(\operatorname{glb}|A_+(\tau)| \operatorname{glb}|A_-(\tau)|\big)^{-1} \|Av, R_+\| \tag{17}$$

is valid for all v *satisfying* (12).

PROOF OF LEMMA 4. Here we use the notation $\|u, R\|_E = \|Eu, R\|$, $\|u, R_+\|_E = \operatorname{glb}\|u_1, R\|_E$, for any convolution operator E. The corresponding spaces are denoted by $H^E(R)$, $H^E(R_+)$. (Note that if $E = A$ or $= A^{-1}$, then $H^E(R)$ and $H^E(R_+)$ depend only on the degree of A.) Evidently

$$\begin{aligned}\|v, R_+\|_{A_+}{}^2 \leqq \|v, R\|_{A_+}{}^2 &= (A_+v, A_+v) = (A_-{}^{-1}Av, A_+v)\\ \leqq \|A_-{}^{-1}Av, R_+\|\,\|A_+v, R_+\| &= \|Av, R_+\|_{A_-^{-1}}\,\|v, R_+\|_{A_+}, \qquad v \in \mathscr{D}(R_+) ,\end{aligned}$$

for in view of the Paley–Wiener theorem

$$\|Eg, R_+\| = \|g, R_+\|_E$$

if $\hat{E}(\tau)$ is regular in the upper half-plane (cf. [9, formula (10)]). Consequently (16) holds for all v in $\mathscr{D}(R_+)$. Since the closure of $\mathscr{D}(R_+)$ in the norm $\|v, R_+\|_{A_+}$ contains all v in R_+ satisfying (12), we see that (16) is also valid for these v. The proof is complete.

PROOF OF COROLLARY. We have obviously

$$\|v, R_+\| \leqq \big(\operatorname{glb}|A_+(\tau)|\big)^{-1} \|v, R_+\|_{A_+}$$

and

$$\|Av, R_+\|_{A_-^{-1}} \leqq \big(\operatorname{glb}|A_-(\tau)|\big)^{-1} \|Av, R_+\| .$$

Hence (17) follows.

2. Estimates in the case $n=1$.

In the previous section we have reduced the problem to the case $n=1$. Accordingly we will now consider ordinary differential operators A with constant coefficients such that $A(\tau)\neq 0$ for τ real, and C^∞ functions v on $\bar{R}_+$ with compact support.

Let $\mathscr{B}$ be any finite dimensional vector space of differential operators with constant coefficients. We consider the following problem:

$$\text{(P)}\qquad \begin{cases} \textit{To find a function } v\in \boldsymbol{H}(R_+) = H^A(R_+) \textit{ such that} \\ (18) \qquad Av = g,\ Bv(0) = 0 \textit{ for all } B\in\mathscr{B}, \\ g \textit{ being a given function } \in H(R_+). \end{cases}$$

We say that (P) is correctly posed if there exists a constant C such that for every g there is a solution v such that

$$(19)\qquad |||v,R_+||| \leqq C\|g,R_+\|,$$

where it will now be convenient to take $|||v,R_+|||$ to be $\|v,R_+\|_A$ (cf. (5)). (Uniqueness is not required.) Clearly the Dirichlet problem is obtained if we take $\mathscr{B}$ to be the set of differential operators of order $<m_+$. Our goal is to evaluate the "best" constant C. The main step will be to reduce (P) to the following equivalent problem (on the whole axis R):

$$(\mathrm{P}_1)\qquad \begin{cases} \textit{To find } \varphi\in H_{R_-} \textit{ such that for some } v_1\in \boldsymbol{H} \\ (20) \qquad Av_1 = g_1+\varphi,\ Bv_1(0) = 0 \textit{ for all } B\in\mathscr{B}, \\ g_1 \textit{ being a given function } \in H_{R_+}. \end{cases}$$

(We note that v_1 is in fact uniquely determined by φ and g_1). We say that (P_1) is correctly posed if for every g_1 there exists a solution φ such that

$$\|\varphi,R\| \leqq C_1\|g_1,R_+\|.$$

Then of course

$$|||v_1,R||| \leqq (1+C_1)\|g_1,R_+\|,$$

by the triangle inequality. Put now $v=v_1\,|\,R_+$ (restriction) and $g=g_1\,|\,R_+$. Then (18) is fulfilled and moreover

$$|||v,R_+||| \leqq (1+C_1)\|g,R_+\|,$$

so that (P) is correctly posed if (P_1) is, and moreover $C\leqq 1+C_1$. Suppose now that (P) is correctly posed. Put again $g=g_1\,|\,R_+$ and let v be a corresponding solution. Extend v to a function v_1 in H. This can evidently be done in such a manner that

$$|||v_1,R||| = |||v,R_+|||.$$

It follows that

$$|||v_1,R||| \leqq C\|g_1,R\|.$$

Put $\varphi = Av_1 - g_1$. Then (20) is of course fulfilled, and we obtain

$$\|\varphi, R\| \leqq (1+C)\|g_1, R\| ,$$

again by the triangle inequality. Hence (P_1) is correctly posed, and $C_1 \leqq 1+C$. To sum up, we have now proved that (P) and (P_1) are equivalent and that moreover C and C_1 are bounded at the same time, as functions of A and $\mathscr{B}$.

Take now Fourier transforms in (20) assuming that (P) is correctly posed. We then obtain

$$A(\tau)\hat{v}_1(\tau) = \hat{g}_1(\tau) + \hat{\varphi}(\tau), \qquad \int B(\tau)\hat{v}_1(\tau)\, d\tau = 0 .$$

Since $A(\tau) \neq 0$ for τ real by assumption, we obtain

$$\int \big(B(\tau)/A(\tau)\big)\big(\hat{g}_1(\tau) + \hat{\varphi}(\tau)\big)\, d\tau = 0 . \tag{21}$$

Conversely, if $\varphi \in H_{R_-}$ satisfies (21), then φ is a solution of (P_1). Put

$$\hat{K}(\tau) = \overline{\big(B(\tau)/A(\tau)\big)} ,$$

and consider K (the inverse Fourier transform of $\hat{K}(\tau)$) as an element of H. Let K_+ and K_- be the projections of K on H_{R_+} and H_{R_-}. We then obtain from (21) the following formula:

$$(K^-, \varphi) = -(K^+, g_1) \qquad \text{for all} \quad B \in \mathscr{B} . \tag{22}$$

It follows from (22) that $K^- = 0$ if and only if $K = 0$ (cf. [1, p. 633]). Choose now a basis $\{B_j\}_{j=1}^l$ in $\mathscr{B}$ such that $\{K_j^-\}_{j=1}^l$ (K_j corresponds to B_j!) is an orthogonal basis. Then φ is of the form

$$\varphi = \sum_{j=1}^{l} -(K_j^+, g_1)/\|K_j^-\|^2 K_j^- + \psi ,$$

where ψ is orthogonal to $\{K_j^-\}_{j=1}^l$. It follows that

$$\sum_{j=1}^{l} |(K_j^+, g_1)|^2/\|K_j^-\|^2 \leqq C_1^2 \|g_1\|^2 . \tag{23}$$

Take $g_1 = K_j^+$. Then we get

$$\|K_j^+\| \leqq C_* \|K_j^-\|$$

with $C_* \leqq C_1$. Since for every $K \neq 0$ evidently K^- is part of an orthogonal basis, we have

$$\|K^+\| \leqq C_* \|K^-\| \qquad \text{for all} \quad B \in \mathscr{B} . \tag{24}$$

Conversely, if (24) holds, then clearly also (23), with $C_1 \leqq lC_*$, and hence (19) hold. Expand now B/A in partial fractions:

$$B(\tau)/A(\tau) = B_+(\tau)/A_+(\tau) + B_-(\tau)/A_-(\tau), \qquad \text{degree } B_\pm < \text{degree } A_\pm .$$

(Here A_+ and A_- are defined as by (15).) Then $K^\pm$ is the inverse Fourier transform of $\left(\overline{B_\mp(\tau)/A_\mp(\tau)}\right)$, for the roots of $A_\mp(\tau)=0$ are in the upper (lower) half-plane so that the Paley–Wiener theorem applies. We have thus proved the following theorem.

THEOREM 1. *The Problem* (P) *is correctly posed if and only if* $B_+=0$ *implies* $B=0$. *Then there exists a constant* C_* *such that we have*

$$\int |B_-(\tau)/A_-(\tau)|^2 \, d\tau \leqq C_*^2 \int |B_+(\tau)/A_+(\tau)|^2 \, d\tau$$

for all $B \in \mathscr{B}$. *Moreover — and this is the point of course — the constants* C *and* C_* *are bounded at the same time.*

In view of the results of Section 1 (in particular Lemma 3), we have the following corollary, which is the main result of this paper.

COROLLARY. *Let* A *be a properly hypoelliptic differential operator with constant coefficients in* R^n *(*n *arbitrary!). Then* (1) *holds for all* u *satisfying* (2) *if and only if the following condition is satisfied: There exists a constant* C_* *such that* (7) *holds for all functions* $B(\xi)$ *such that* $B(\xi)$ *is a polynomial in* τ *of degree* $<m_+$ $\big(B_+(\xi)$ *and* $B_-(\xi)$ *being defined as by* (6)$\big)$, *and all* ξ' *outside some compact set.*

REMARK. Note that there is a formal analogy between the definition of elliptic boundary problems, as given e.g. in [1, p. 633], and our condition (7). In [1] it is required that the boundary operators $B_j(\xi', \tau)$ should be independent modulo $A_+(\xi', \tau)$ for every ξ', while as (7) in a sense expresses that this independence should be "uniform".

3. The special operator $(D_t - i\Delta^2)(D_t + i\Delta)$.

We are now going to study the operator

$$A = (D_t - i\Delta^2)(D_t + i\Delta)$$

and we will first give a direct proof, which does not utilize the results of the previous sections, that the inequality

$$\|A_1 u, R_+^n\| \leqq C(\|Au, R_+^n\| + \|u, R_+^n\|) , \tag{25}$$

where

$$A_1 = (D_t + i\Delta^2)(D_t - i\Delta) ,$$

cannot hold true. (Note that A_1 is not the same operator as in the Introduction!) First we observe that (25) is equivalent to

$$\|A_1(\xi', D_t)v, R_+\| \leqq C\big(\|A(\xi', D_t)v, R_+\| + \|v \,.\, R_+\|\big); \tag{26}$$

this may be proved by an argument similar to the one used in the proof of Lemma 1. Take now

$$v(t) = e^{-t|\xi'|^2} - e^{-t|\xi'|^4}.$$

Then v satisfies (2) and hence we may apply (26) to it. Since

$$A_1 v = |\xi'|^6 (|\xi'|^2 - 1)\, e^{-t|\xi'|^4}$$

and

$$Av = |\xi'|^4 (|\xi'|^2 - 1)\, e^{-t|\xi'|^2},$$

we obtain

$$|\xi'|^4 \big||\xi'|^2 - 1\big| \leqq C\big(|\xi'|^3 \big||\xi'|^2 - 1\big| + |\xi'|^{-1} + |\xi'|^{-2}\big),$$

which leads to a contradiction if we let ξ' tend to ∞.

Next we consider the case when the constant C in (25) depends on the diameter of the support of u. We will see that also in this case (25) cannot hold true. In fact, using a partition of unity, it is easily seen that the following inequality holds

$$\|A_1 u, R^n{}_+\| \leqq C\big(\|Au, R^n{}_+\| + \|D_t u, R^n{}_+\|_3 + \|u, R^n{}_+\|_5\big)$$

for all u satisfying (2). (No restriction on the diameter of the support of u.) Using the above construction one easily verifies that also this inequality leads to a contradiction. We leave the details to the reader.

4. Estimates for variable coefficients and regularity.

Let $A = A(D)$ be properly hypoelliptic, and assume that there is a constant C such that the following inequality

$$|||\hat{u}(\xi', .), R_+|||_{\xi'} \leqq C\big(\|A\hat{u}(\xi', .), R_+\| + \|\hat{u}(\xi', .), R_+\|\big) \tag{27}$$

holds for all u satisfying (2) and all ξ'. (In view of the results of Section 1 and Section 2, this is of course equivalent to the requirement that (7) should hold.) Put

$$E = (1 + \Delta)^{\frac{1}{2}}.$$

Multiply the square of (27) by $\big(E(\xi')\big)^{2r} d\xi'$ where

$$E(\xi') = (1 + |\xi'|^2)^{\frac{1}{2}}$$

and r is any real number, and integrate. We obtain

$$\int |||\big(E(\xi')\big)^r \hat{u}(\xi', .), R_+|||_{\xi'}^2 \, d\xi' \leqq C^2 \Big(\int \big\|\big(E(\xi')\big)^r A u(\xi', .), R_+\big\|^2 d\xi' + \int \big\|\big(E(\xi')\big)^r \hat{u}(\xi', .), R_+\big\|^2 d\xi'\Big)$$

or, in view of formula (10),

$$|||u, R^n{}_+|||_r \leqq C(\|Au, R^n{}_+\|_r + \|u, R^n{}_+\|_r) .$$

Here we use the norms:

$$|||u, R^n{}_+|||_r = |||E^r u, R^n{}_+||| \tag{28}$$

and

$$\|u, R^n{}_+\|_r = \|E^r u, R^n{}_+\| . \tag{29}$$

Since

$$\|u, R^n{}_+\|_r \leqq C|||u, R^n{}_+|||_{r-w}$$

for some $w>0$, in view of the hypoellipticity, the remainder can be replaced by $|||u, R^n{}_+|||_{r-w}$ and in fact even by $|||u, R^n{}_+|||_{r-1}$ for we have an inequality of Ehrling–Nirenberg type:

$$|||u, R^n{}_+|||_{r-w} \leqq \varepsilon|||u, R^n{}_+|||_r + C_\varepsilon|||u, R^n{}_+|||_{r-1}, \qquad \varepsilon>0 .$$

Hence we obtain the following inequality:

$$|||u, R^n{}_+|||_r \leqq C(\|Au, R^n{}_+\|_r + |||u, R^n{}_+|||_{r-1}) . \tag{30}$$

We want now to extend (30) to variable coefficients. We consider accordingly differential operators of the form (cf. [9], [10]):

$$A = A(x,D) = A_0(D) + \sum_{k=1}^{s} c_k(x) A_k(D), \; c_k(0) = 0 , \tag{31}$$

where 1° A_0 is properly hypoelliptic satisfying the inequality (27), and A_k, $k=1,2,\ldots,s$, are weaker than A_0 in the sense of [3], 2° c_k, $k=1,2,\ldots,s$, are C^∞ functions with compact support such that the quantity

$$\delta = \sum_{k=1}^{s} \sup |c_k(x)|$$

is smaller than a positive constant δ_0 to be determined, depending on A_k, $k=0,1,\ldots,s$, only. Clearly every operator of the form (31) is hypoelliptic in the vicinity of 0, and conversely every hypoelliptic operator is of the form (31) in the vicinity of 0. It follows from 1° that we have

$$|||u, R^n{}_+|||_r \leqq C_0(\|A_0 u, R^n{}_+\|_r + |||u, R^n{}_+|||_{r-1}) \tag{32}$$

and

$$\|A_k u, R^n{}_+\|_r \leqq C_k|||u, R^n{}_+|||_r . \tag{33}$$

($|||\cdot, R^n{}_+|||_r$ is of course the norm corresponding to A_0.) Now, by the triangle inequality, we obtain

$$\|A_0 u, R^n{}_+\|_r \leqq \|Au, R^n{}_+\|_r + \sum_{k=1}^{s} \|c_k A_k u, R^n{}_+\|_r . \tag{34}$$

We have now the following lemma.

LEMMA 5. *Let c be any C^∞ function with constant support and let r be any real number. There exists a constant Γ such that*

$$\|cv, R^n{}_+\|_r \leqq \sup|c(x)|\, \|v, R^n{}_+\|_r + \Gamma\|v, R^n{}_+\|_{r-1}\,.$$

REMARK. This is a variant of a lemma by Hörmander and Lions [6]. For the proof see [9], [10].

Applying Lemma 5 we obtain from (32), (33), (34):

$$|||u, R^n{}_+|||_r \leqq C_0\|Au, R^n{}_+\|_r + C_0|||u, R^n{}_+|||_{r-1} + \\ + \sum C_0 C_k \sup|c_k(x)|\, |||u, R^n{}_+|||_r + \sum C_0 C_k \Gamma_k |||u, R^n{}_+|||_{r-1}\,.$$

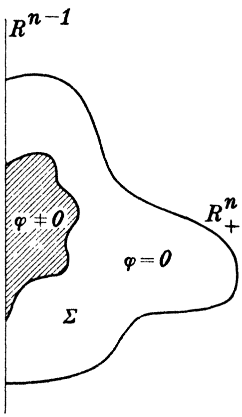
Hence, if δ is sufficiently small (30) follows. Thus we have the following theorem.

THEOREM 2. *If A is of the form* (31) *and δ is sufficiently small, then the inequality* (30) *holds for all u satisfying* (2).

Next, let $\boldsymbol{H}^r(R^n{}_+)$ and $H^r(R^n{}_+)$ be the spaces corresponding to the norms (28) and (29) respectively. Then we have

THEOREM 3. *If $u \in \boldsymbol{H}^q(R^n{}_+)$ for some q and satisfies* (2) *and if $Au \in H^r(R^n{}_+)$, then $u \in \boldsymbol{H}^r(R^n{}_+)$.*

PROOF. Suppose that $q \leqq r-1$. Apply (30) with q instead of r, to $(u_{h'} - u)/|h'|$, $h' \in R^{n-1}$ where $u_{h'}(x) = u(x'+h', t)$. It is then easy to see that the right hand side and consequently the left hand side will remain bounded as h' tends to 0. By weak compactness it follows now in a well-known manner that actually $u \in H^{q+1}(R^n{}_+)$, so that q can be replaced by $q+1$ and hence, after a finite number of steps, by r. This proves the theorem.



Let Σ be an open set in $R^n{}_+$. It is supposed that every point on the intersection of R^{n-1} and the boundary of Σ is bounded away from the rest of the boundary. Let $H^r_{\text{loc}}(\Sigma)$ be the space of distributions u on Σ such that $\varphi u \in H^r(R^n{}_+)$ for all C^∞ functions φ with the support contained in a compact set of $\Sigma^* = \Sigma \cup (\bar{\Sigma} \cap R^{n-1})$. In a similar manner $\boldsymbol{H}^r_{\text{loc}}(\Sigma)$ is defined.

THEOREM 4. *If $u \in \mathscr{D}'(\Sigma^*)$ and satisfies* (2) *on $(\bar{\Sigma} - \Sigma) \cap R^{n-1}$ and if $Au \in H^r_{\text{loc}}(\Sigma)$, then $u \in \boldsymbol{H}^r_{\text{loc}}(\Sigma)$.*

PROOF. By partial regularity (cf. [8], [9]), we may assume that

$u \in \boldsymbol{H}^q_{\text{loc}}(\Sigma)$ for some q. Then $\varphi u \in \boldsymbol{H}^q(R^n{}_+)$ and satisfies also (2), and $A\varphi u \in H^{\text{glb}(q+z,r)}(R^n{}_+)$ (by Leibniz's formula) for some $z > 0$. For since A_0 is hypoelliptic, it follows that

$$|A_k{}^\alpha(\xi)| \leqq C(1+|\xi'|^2)^{-\frac{1}{2}z}(|A_0(\xi)|+1), \qquad \alpha \neq 0\,,$$

for some $z > 0$, so that

$$\|A_k{}^\alpha u, R^n{}_+\|_{q+z} \leqq C\,|||u, R^n{}_+|||_q\,.$$

So Theorem 3 shows that $\varphi u \in \boldsymbol{H}^{\text{glb}(q+z,r)}(R^n{}_+)$ which means that $u \in \boldsymbol{H}^{\text{glb}(q+z,r)}_{\text{loc}}(\Sigma)$. In other words, q can be replaced by $\text{glb}(q+z,r)$, so that after a finite number of steps we obtain r. This proves the theorem.

Corollary. *If $u \in \mathscr{D}'(\Sigma^*)$ and satisfies* (2) *and if $Au \in \mathscr{E}(\Sigma^*)$, then $u \in \mathscr{E}(\Sigma^*)$.*

Proof. By partial regularity.

Remarks. 1° To sum up we have proved the regularity near a plane portion of the boundary of the solutions of the Dirichlet problem for properly hypoelliptic operators. A satisfying the condition (7). Occasionally one can also extend this to the case when the whole or part of the boundary is curved, as in the elliptic and, more generally, the parabolic case. For symmetric operators Malgrange [7] extended Gårding's inequality [2] for elliptic operators to hypoelliptic operators. It follows that the Dirichlet problem has a weak solution. Our results show that the solution is a strong one, i.e. regular in any domain that is bounded by a finite number of planes, except in the vicinity of the intersection of these planes, where no regularity can be expected in general.

2° The corollary of Theorem 4 is probably also true without any hypothesis of the type (7), at least for some operators. One has then to use estimates derived from Lemma 4. The difficulty is to extend Lemma 5 to these new norms.

5. Various extensions and remarks.

As the reader may have remarked the hypoellipticity plays a minor rôle throughout the discussion (except in Section 4 where it is indispensable). In fact it is easy to formulate the main result (Corollary of Theorem 1) for a much bigger class of operators.

Also the restriction to Dirichlet boundary conditions has been mostly done for convenience only. The hard point is to extend the Corollary of Lemma 3. But under fairly general assumptions we have

$$(E(\xi'))^p\,|||\hat{u}(\xi',.), R_+|||_{\xi'} \leqq C\,\|\widehat{Au}(\xi',.), R_+\|$$

for all ξ' outside some compact set S and some real number p, u being any C^∞ function with compact support in $R^n{}_+$ satisfying the condition:

$$B_j u = 0, \qquad j = 1,2,\ldots,m_+ ,$$

on R^{n-1}, and this suffices to carry through the reduction to the case $n=1$.

One may also consider inequalities corresponding to non-homogeneous boundary conditions. Let us outline a deduction in the case of ordinary operators, the notation being that of Section 2. Instead of (21) we have now the formula:

$$\int \big(B(\tau)/A(\tau)\big)\big(\hat{g}_1(\tau)+\hat{\varphi}(\tau)\big)\,d\tau = c ,$$

c being the prescribed value of $Bu(0)$, which yields

$$(K_j^-,\varphi) = -(K_j^+,g_1) + c_j$$

and finally

$$\varphi = \sum_{j=1}^{l}\big(-(K_j^+,g_1)+\bar{c}_j\big)/\|K_j^-\|^2 K_j^- .$$

In virtue of (24) we now obtain

$$\|\varphi\| \leqq lC^*\|g_1\| + (1+C^*)\sum_{j=1}^{l}|c_j|/\|K_j\| ,$$

which aparently implies:

$$|||u,R_+||| \leqq C\|Au,R_+\| + C'\sum_{j=1}^{l}|B_j u(0)/\|K_j\| .$$

From this inequality it is now easy to get to the case $n>1$. However, the assumption that $\{K_j^-\}_{j=1}^{l}$ is an orthogonal basis turns out to be highly unnatural; to get an inequality corresponding to a given basis $\{B_j\}_{j=1}^{l}$ in $\mathscr{B}$ one has to impose extra requirements.

Let us finally point out an interesting class of operators which includes in particular the elliptic and more generally the parabolic case. For simplicity, we consider constant coefficients only. Write A in the form

$$A = A(D) = D_t^m + A_{m-1}(D_{x'})D_t^{m-1} + \ldots + A_0(D_{x'}) .$$

Then the defining relation is

$$|A(\xi)| \sim |\tau|^m + |A_0(\xi')|, \quad \xi' \to \infty \tag{35}$$

(i.e. the ratio between the two expressions should be bounded when ξ' lies outside some compact set S). An equivalent formulation is

$$\operatorname*{glb}_r |\operatorname{Im}\tau_r(\xi')| \sim \operatorname*{lub}_r |\tau_r(\xi')|, \quad \xi' \to \infty . \tag{36}$$

Obviously both (35) and (36) are satisfied if A is elliptic or more generally parabolic. For this class one can also successfully treat boundary problems other than the Dirichlet boundary problem. In fact, most of the results and methods of [9, Chap. I], are directly applicable to this case. Only one has to operate with $|A_0(\xi')|^{1/m}$ instead of $(1+|\xi'|^2)^{\frac{1}{2}}$, even in the definition of the relevant norms and spaces.

REMARK. As in the elliptic case it is easy to see directly that (7) is fulfilled.

REFERENCES

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions* I, Comm. Pure Appl. Math. 12 (1959), 623–727.
2. L. Gårding, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. 1 (1953), 55–72.
3. L. Hörmander, *On the theory of general partial differential operators*, Acta Math. 94 (1955), 161–248.
4. L. Hörmander, *On the interior regularity of the solutions of partial differential equations*, Comm. Pure Appl. Math. 11 (1958), 197–218.
5. L. Hörmander, *On the regularity of the solutions of boundary problems*, Acta Math. 99 (1958), 225–264.
6. L. Hörmander and J. L. Lions, *Sur la complétion par rapport à une intégrale de Dirichlet*, Math. Scand. 4 (1956), 259–270.
7. B. Malgrange, *Sur une classe d'opérateurs différentiels hypoelliptiques*, Bull. Soc. Math. France 85 (1957), 283–306.
8. J. Peetre, *Théorèmes de régularité pour quelques classes d'opérateurs différentiels*, Thesis, Lund, 1959 ($\approx$ Med. Lunds Univ. Mat. Sem. 16 (1959), 1–122).
9. J. Peetre, *Another approach to elliptic boundary problems*, To appear in Comm. Pure Appl. Math.
10. J. Peetre, *A proof of the hypoellipticity of formally hypoelliptic differential operators*, To appear in Comm. Pure Appl. Math.
11. M. Schechter, *Solution of the Dirichlet problem for systems not necessarily strongly elliptic*, Comm. Pure Appl. Math. 12 (1959), 241–247.
12. V. Thomée, *Locally cogent boundary operators*, Math. Scand. 7 (1959), 5–32.