A PROBLEM FOR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS DEFINED ON AN ARBITRARY OPEN SET

N. O. WALLIN

1. Introduction.

Let \( \Omega \) be an arbitrary open set in real \( n \)-space \( \mathbb{R}^n \) and denote by \( H = H(\Omega) \) the Hilbert space of all square integrable functions over \( \Omega \). Let
\[
\Lambda = (\partial/\partial x_1)^2 + \ldots + (\partial/\partial x_n)^2
\]
be Laplace’s operator and \( V = V(x_1, \ldots x_n) \) a real potential defined in \( \Omega \). Consider the Schrödinger operator
\[
A = -\Lambda + V
\]
with domain of definition consisting of all functions \( f \) in \( H \) such that \( Af \) in the distribution sense is also a function in \( H \). If \( \Omega = \mathbb{R}^n \) it was shown by Carleman that \( A \) is self-adjoint provided that \( V \) is continuous and bounded from below [2]. Recently this was generalized by Browder [1], who in our terminology replaced \(-\Lambda\) by a differential operator \( a \) with uniformly elliptic and positive principal part and with suitably bounded coefficients. He proved that
\[
(a + V)^* = (\bar{a} + V)
\]
provided that \( V \) is continuous and bounded from below and \( \Omega = \mathbb{R}^n \). In (1.1) both operators have their maximal domains of definition. The asterisk denotes the adjoint in \( H \) and \( \bar{a} \) is the formal adjoint of \( a \). In this paper a similar problem is treated. We shall replace \(-\Lambda\) by a differential operator \( a \) which is defined in \( \Omega \) and has an elliptic and positive principal part there. The coefficients of \( a \) are assumed to possess continuous derivatives up to a certain order but need not be bounded in \( \Omega \) and the ellipticity of \( a \) may be non-uniform. Then we shall show the following theorem:

**Theorem.** (1.1) holds for every continuous potential \( V \) that increases rapidly enough at the boundary of \( \Omega \).

Received April 5, 1961.
The rate of increase of $V$ depends of course on the operator $a$. If $\Omega = R^n$ and the ellipticity of $a$ is uniform and if suitable conditions of boundedness are imposed on the coefficients of $a$ one can obtain Browder’s result in a less general form. This case is not treated here. Our theorem also holds when $\Omega$ is a differentiable manifold. The proof is similar and will not be given.

The proof of our theorem proceeds in two steps. The order of $a$ is necessarily even, say $2m$. Denote by $D_\alpha f$ the derivatives of $f$ and let $|\alpha|$ be the order of $D_\alpha f$. Put
\[
|D^f f(x)|^2 = \sum_{|\alpha| \leq j} |D_\alpha f(x)|^2.
\]
In the first step we show that
\[
(1.2) \quad \int \left( \epsilon(x) |D^m f(x)|^2 + r(x) |D^{m-1} f(x)|^2 \right) dx < \infty
\]
for any $f$ in the domain of $a + V$ og $\bar{a} + V$. Here $\epsilon = \epsilon(x)$ and $r = r(x)$ are positive functions, $\epsilon$ depending on the ellipticity of $a$ and $r$ on the potential $V$. An essential point is that $r$ can be taken arbitrarily large provided that $V$ is large enough. To prove the theorem it is sufficient to show that
\[
(1.3) \quad ((a + V)f, g) = (f, (\bar{a} + V)g)
\]
for every $f$ and $g$ in the domains of $a + V$ and $\bar{a} + V$, respectively (i.e. the inclusion $(\bar{a} + V) \subset (a + V)^*$). The scalar product in (1.3) is that of $H$. To prove (1.3) we multiply the integrands of the two sides by a suitable infinitely differentiable function $\psi = \psi(x)$ which is equal to 1 on a large compact part of $\Omega$ and vanishes in a neighbourhood of the boundary of $\Omega$. Forming the difference between the two integrals and integrating by parts one gets an expression which by (1.2) tends to zero as $\psi$ tends to 1 in all of $\Omega$. This is the second step.

In the proof of (1.2) we use a number of well known inequalities plus a new one which may be useful in various situations. Consider an integral of the form
\[
(1.4) \quad L(f) = L_{m, m-1}(f, f) = \int_{\Omega} \sum l_{\alpha \beta}(\varphi) \, D_\alpha f \overline{D_\beta f} \, dx ,
\]
where the summation extends over $|\alpha| \leq m$ and $|\beta| \leq m - 1$. The function $f$ is assumed to possess locally square integrable weak derivatives $D_\alpha f$ for all $\alpha$ such that $|\alpha| \leq m$. About $\varphi$ we assume: $\varphi$ is infinitely differentiable with compact support in $\Omega$ and the function
\[ \varphi^* = \frac{D_{\alpha_1} \varphi \ldots D_{\alpha_k} \varphi}{q^{k-1}} \]

has the same property for every integer \( k \geq 1 \). The coefficients \( l_{\alpha \beta}(\varphi) \) are assumed to be quadratic forms in the functions \( \varphi^* \) with coefficients in \( C^{|\alpha|+|\beta|}(\Omega) \) where \( C^k(\Omega) \) is the set of functions with continuous derivatives \( D_\alpha f \) for all \( \alpha \) such that \( |\alpha| \leq k \). Then, denoting by \( S \varphi \) the support of \( \varphi \),

\[ |L(f)| \leq s \int_{S \varphi} |D^m qf|^2 \, dx + t \int_{S \varphi} |f|^2 \, dx + \tau \int_{S \varphi} |qf|^2 \, dx, \]

where \( s \) and \( t \) are positive numbers which may be chosen arbitrarily small provided that \( \tau = \tau(s,t,q,(l_{\alpha \beta})) \) is taken large enough.

2. Notations and remarks.

We consider a linear elliptic differential operator \( a \) of order \( 2m \) defined for every \( x \) in \( \Omega \) by the expression

\[ a = a(x,D) = \sum_{|\alpha| \leq 2m} a_{\alpha}(x) D_\alpha, \]

where \( \alpha = (\alpha_1, \alpha_2 \ldots \alpha_j) \), with \( 1 \leq \alpha_k \leq n \) and \( |\alpha| = j \leq 2m \). Furthermore

\[ D_{\alpha_1} = \partial \! /\! / \partial x_{\alpha_1}, \quad D_\alpha = D_{\alpha_1} \ldots D_{\alpha_j}. \]

The formal adjoint \( \overline{a} \) of \( a \) is by definition

\[ \overline{a} = \overline{a}(x,D) = \sum_{|\alpha| \leq 2m} D_\alpha a_{\alpha}(x). \]

Since the operator is supposed to be elliptic

\[ \text{Re} a_0(x, \xi) = \text{Re} \sum_{|\alpha| = 2m} a_{\alpha}(x) \xi_\alpha, \quad \xi_\alpha = \xi_{\alpha_1} \ldots \xi_{\alpha_{2m}}, \]

is a definite form in \( \xi \) for all \( x \) in \( \Omega \). Let it be positive. Then

\[ \text{Re} a_0(x, \xi) \geq g(x) \sum_{|\alpha| = m} \xi_\alpha^2, \]

where \( g(x) > 0 \) on \( \Omega \). If the coefficients of \( a_0 \) are continuous, \( g(x) \) may be chosen to be continuous. Since \( \text{Re} \overline{a}_0(x, \xi) = \text{Re} a_0(x, \xi) \) the same is true for \( \overline{a} \).

By \( C^k(\Omega) \) we shall denote, as before, the space of \( k \) times continuously differentiable functions in \( \Omega \). The space of infinitely differentiable functions in \( \Omega \) is denoted by \( C^\infty(\Omega) \), and \( C^\infty_0(\Omega) \) is the set of functions in \( C^\infty(\Omega) \) with compact supports. As in the introduction \( H = H(\Omega) \) is the Hilbert space of all square integrable functions in \( \Omega \). The set of functions
f such that \(D_\alpha f\) in the weak sense is square integrable on every compact subset of \(\Omega\) for all \(|x| \leq k\) is denoted by \(\mathcal{H}^k(\Omega)\). The corresponding set of functions with compact supports is \(\mathcal{H}^k_0(\Omega)\). It is well known that for elliptic operators \(A\) of order \(2m\) with sufficiently differentiable coefficients in \(\Omega\), \(Af = g\) implies that \(f\) is in \(\mathcal{H}^{2m}(\Omega)\) if \(f\) and \(g\) are in \(\mathcal{H}(\Omega)\), cf. [4]. We shall assume that \(a\) and \(\bar{a}\) fulfill this regularity condition.

3. Some inequalities.

Let us introduce the following notation:

\[
|D_\alpha f, S|^2 = \int_S |D_\alpha f(x)|^2 \, dx,
\]

\[
|D^k f(x)|^2 = \sum_{|\alpha| \leq k} |D_\alpha f(x)|^2,
\]

\[
|D^k f, S|^2 = \int_S |D^k f(x)|^2 \, dx.
\]

If \(S = \Omega\) we write simply \(|D^k f|^2\) instead of \(|D^k f, \Omega|^2\) etc.

We shall show that with a suitable countable locally finite covering of \(\Omega\) by closed spheres \((S_k)\) and a suitable partition of unity \(1 = \sum \varphi_k^2\) belonging to that covering the following inequalities are valid,

(i):

\[
\text{Re}(af, f) \geq \varepsilon_k |D^m f|^2 - \tau_k |f|^2
\]

for all \(f \in \mathcal{H}^m_0(S_k)\). Here \(\varepsilon_k\) and \(\tau_k\) are positive numbers that depend on the ellipticity and the coefficients of the operator \(a\) in \(S_k\).

(ii):

Let \(f \in \mathcal{H}^m(\Omega)\) and denote by \(\varphi\) an arbitrary \(\varphi_k\). If

\[
L_{m, m-1}(f, f) = \int \sum_{|\alpha| \leq m} l_{\alpha \beta}(\varphi) \, D_\alpha f \overline{D_\beta f} \, dx,
\]

where the functions \(l_{\alpha \beta}(\varphi)\) are the same as in (1.4), then

\[
|L_{m, m-1}(f, f)| \leq s |D^m qf|^2 + \tau(s, t, \varphi, l)|qf|^2 + t|f, St \varphi|^2.
\]

Here \(s\) and \(t\) are arbitrary positive numbers and \(\tau(s, t, \varphi, l)\) is a positive number that depends on \(s, t, \varphi\) and \(l\) where \(l\) stands for the collection \((l_{\alpha \beta}(\varphi))\). \(St \varphi\) denotes the support of \(\varphi\).

(iii):

For \(f \in \mathcal{H}^m_0(\Omega)\) we have
\[ |D^{m-1}f|^2 \leq \varepsilon |D^m f|^2 + \tau(\varepsilon) |f|^2, \]

where \( \varepsilon > 0 \) is arbitrary and \( \tau(\varepsilon) \) is a positive number such that \( \tau(\varepsilon) \to \infty \) as \( \varepsilon \to 0. \)

(iv):

Let \( f \in \mathcal{H}^{2m}(\Omega) \). Then

\[ \text{Re}(q_k^2af,f) \geq \varepsilon_k |q_k D^m f|^2 + r_k |q_k D^{m-1} f|^2 - t_k |f, \text{St} q_k|^2 - \tau(r_k,t_k) |q_k f|^2. \]

In this inequality \( \varepsilon_k \) is a positive number that depends on the ellipticity of the operator in \( S_k \), \( r_k \) and \( t_k \) are arbitrary positive numbers and \( \tau(r_k,t_k) \) is a positive number that depends on \( r_k \) and \( t_k \).

(v):

If \( (\tau_k) \) is a sequence of positive numbers there exists a function \( \tau(\tau) \) in e.g. \( C^\infty(\Omega) \) such that

\[ \min_{x \in S_k} \tau(x) \geq \tau_k. \]

Now we take \( t_k = 2^{-k} \) in (iv) and majorize \( \tau_k = \tau(r_k, 2^{-k}) \) by \( V_r = V(r,x) \) according to (v). The number \( r \) indicates the dependence on the sequence \( (r_k) \). This results in

(iv'):

\[ \text{Re}(q_k^2(a+V_r)f,f) + 2^{-k} |f, \text{St} q_k|^2 \geq \varepsilon_k |q_k D^m f|^2 + r_k |q_k D^{m-1} f|^2. \]

If \( (a+V_r)f \in H \) and \( f \in H \) then, since \( V_r \in C^0(\Omega), f \in \mathcal{H}^{2m}(\Omega) \). This is a consequence of the regularity theorem for elliptic operators. From (iv') and the fact that \( |f, \text{St} q_k| \leq |f| \) and \( \sum 2^{-k} = 1 \) it follows by Lebesgue's theorem that

(vi):

\[ \text{Re}((a+V_r)f,f) + |f|^2 \geq \int_{\Omega} \varepsilon(x) |D^m f(x)|^2 \, dx + \int_{\Omega} r(x) |D^{m-1} f(x)|^2 \, dx, \]

i.e.

\[ \int_{\Omega} \varepsilon(x) |D^m f(x)|^2 \, dx < \infty \]

and

\[ \int_{\Omega} r(x) |D^{m-1} f(x)|^2 \, dx < \infty, \]

where \( 0 < \varepsilon(x) = \sum \varepsilon_k q_k^2(x) \) and \( 0 < r(x) = \sum r_k q_k^2(x) \). It is easy to see that the inequality (vi) is also true with the same \( \varepsilon(x), r(x) \) and \( V_r \) if \( a + V_r \).
is replaced by $\bar{a} + V_r$ provided that $V_r$ is any continuous function $\geq \max(V_r(a, x), \text{ where } V_r(a, x))$ is, for example $V_r = V_r(a, x) + V_r(\bar{a}, x)$. Here $V_r(a, x)$ refers to the potential constructed in (iv').

4. Proof of the theorem.

We are now in a position to prove the inclusion $(a + V_r) \supseteq \bar{a} + V_r$ (the reverse one is trivial) for a suitable $V_r$. Therefore let $(\Omega_i)$ be a compact covering of $\Omega$ such that $\Omega_{i-1} \subseteq \text{int}(\Omega_i)$ and introduce the function $\psi_i \in C_0^\infty(\Omega_i), 0 \leq \psi_i \leq 1$ with $\psi_i = 1$ on $\Omega_{i-1}$. By the definition of $\bar{a}$,

$$(\psi_i(a + V_r)f, g) = (f, (\bar{a} + V_r)\psi_i g)$$

for every $f$ and $g$ in the domains of $a + V_r$ and $\bar{a} + V_r$ respectively. Since

$$\bar{a} \psi_i g = \psi_i \bar{a} g + \sum_{|\alpha| \leq 2m-1} b_\alpha(\psi_i) D_\alpha g,$$

where $b_\alpha(\psi_i)$ is a linear combination of derivatives of $\psi_i$, it follows that

$$(\psi_i(a + V_r)f, g) = (\psi_i f, (\bar{a} + V_r)g) + \sum_{|\alpha| \leq 2m-1} (f, b_\alpha(\psi_i) D_\alpha g).$$

The second term on the right hand side may be transformed by partial integrations into

$$L_{m, m-1}^{(i)}(f, g) = \sum_{|\alpha| \leq m} \int_{\omega_i} l_{\alpha \beta}(\psi_i) D_\alpha f \overline{D_\beta g} \, dx$$

the integrations being performed over $\omega_i = \Omega_i - \Omega_{i-1}$. With

$$\sup_{a, \beta, x} |l_{\alpha \beta}(\psi_i)| = C_i$$

we obtain

$$|L_{m, m-1}^{(i)}(f, g)| \leq \sum_{|\alpha| \leq m} \int_{\omega_i} C_i \, |D_\alpha f(x)| \, |\overline{D_\beta g(x)}| \, dx$$

$$\leq \sum_{|\alpha| \leq m} \int_{\omega_i} \varepsilon(x) |D_\alpha f(x)| \frac{C_i}{\varepsilon(x)^{\frac{1}{4}}} \, |D_\beta g(x)| \, dx$$

$$\leq \sum_{|\alpha| \leq m} \int_{\omega_i} \left( \varepsilon(x) |D_\alpha f(x)|^2 + \frac{C_i^2}{\varepsilon(x)} |D_\beta g(x)|^2 \right) \, dx$$

$$\leq d_{m-1} \int_{\omega_i} \varepsilon(x) |D^m f(x)|^2 \, dx + d_m \int_{\omega_i} \frac{C_i^2}{\varepsilon'} |D^{m-1} g(x)|^2 \, dx,$$

where $\varepsilon' = \inf_{\omega_i} \varepsilon(x) > 0$ and $d_k$ is the number of derivatives of order
\( \leq k \). Let \( \chi_i(x) = 1 \) on \( \omega_i \) and zero elsewhere be the characteristic function of \( \omega_i \) and put

\[
\chi(x) = \sum C_i^2 \varepsilon_i \chi_i(x) .
\]

This expression is well defined for every \( x \in \Omega \) since the covering \( (\omega_i) \) where \( \omega_1 = \Omega_1 \) is locally finite. The compactness of \( S_k \) then implies that only a finite number of the \( \omega_i \) will meet \( S_k \) so that

\[
\max_{x \in S_k} \chi(x) = C'_k < \infty .
\]

If \( r_k \geq C'_k \) it follows that

\[
\chi(x) = \sum q_k^2(x) \chi(x) \leq \sum q_k^2(x) C'_k \leq \sum q_k^2(x) r_k = r(x) .
\]

In combination with the fact that

\[
\chi(x) = \frac{C_i^2}{\varepsilon_i} \quad \text{on} \quad \omega_i ,
\]

this choice of \( r(x) \) gives

\[
|L_{m,m-1}^{(i)}(f,g)| \leq d_{m-1} \int_{\omega_i} \varepsilon(x) |D^m f(x)|^2 \, dx + d_m \int_{\omega_i} r(x) |D^{m-1} g(x)|^2 \, dx .
\]

If we let \( i \to \infty \) it follows from (vi) that

\[
\int_{\omega_i} \varepsilon(x) |D^m f(x)|^2 \, dx \to 0 \quad \text{and} \quad \int_{\omega_i} r(x) |D^{m-1} g(x)|^2 \, dx \to 0 ,
\]

that is \( L_{m,m-1}^{(i)}(f,g) \to 0 \). Further \( \psi_i \not\to 1 \) on all of \( \Omega \), and we get by Lebesgue’s theorem

\[
((a + V_r)f,g) = (f,(\bar{a} + V_r)g) .
\]

This proves our assertion.

5. Partition of unity.

Let \( (S_k) \) be a countable locally finite covering of \( \Omega \) by closed spheres \( S_k \) with radius \( r_k \) and center \( z_k \) such that \( S_k \subset \Omega \) for all \( k \). We shall assume that if \( S'_k \) is the closed sphere with center \( z_k \) and radius \( \frac{1}{2} r_k \) the collection \( (S'_k) \) is also a covering of \( \Omega \). It is not difficult but somewhat tedious to show that such coverings exist and we do not go into the details. We are now going to construct a partition of unity belonging to the covering \( (S_k) \), that is, functions \( q_i(x) \) with the following properties:
(5.1) \( 0 \leq \varphi_i(x) \in C^\infty_0(S_i) \),
(5.2) \( 1 = \sum \varphi_i^2(x) \quad \text{on} \quad \Omega \).

To do this put
\[
\gamma_i(x) = \begin{cases} 
\exp \left\{ \frac{r_i^2}{r_i^2 - |x-z_i|^2} \right\} & \text{for} \quad |x-z_i| < r_i , \\
0 & \text{for} \quad |x-z_i| \geq r_i . 
\end{cases}
\]

Then \( \gamma_i(x) \in C^\infty_0(S_i) \) and since \( \gamma_i > 0 \) on \( S'_i \) and \( (S'_k) \) is a covering of \( \Omega \) we have
\[
\sum \gamma_i^2(x) > 0 \quad \text{on} \quad \Omega .
\]

From this it follows that
(5.3) \( \varphi_i(x) = \frac{\gamma_i(x)}{(\sum \gamma_i^2(x))^{\frac{1}{2}}} \in C^\infty_0(S_i) \)

and
\[
1 = \sum \varphi_i^2(x) \quad \text{on} \quad \Omega .
\]

It is easily seen that the function
\[
(5.4) \quad \begin{cases} 
D_{\alpha_1} \gamma_i \cdots D_{\alpha_k} \gamma_i & \text{for} \quad |x-z_i| < r_i , \\
\frac{\gamma_i^{k-1}}{\gamma_i} & \text{for} \quad |x-z_i| \geq r_i . 
\end{cases}
\]
is in \( C^\infty_0(S_i) \) for every integer \( k \geq 1 \). Here \( \alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^k \) are multi-indices. Because of (5.3) the same will be true if we replace \( \gamma_i \) by \( \varphi_i \), i.e.

(5.5) \[
\begin{cases} 
D_{\alpha_1} \varphi_i \cdots D_{\alpha_k} \varphi_i & \text{for} \quad |x-z_i| < r_i , \\
\frac{\varphi_i^{k-1}}{\varphi_i} & \text{for} \quad |x-z_i| \geq r_i . 
\end{cases}
\]
is in \( C^\infty_0(S_i) \). At last we point out that by squaring (5.5) and dividing by \( \varphi_i \) we obtain a new function of the same kind. We also remark that differentiation of (5.5) again leads to a function of the same kind.

6. Proof of the inequalities.

The inequalities in section 3 will now be proved one by one.

(i):

Since \( \varrho(x) \) is a continuous function in \( \Omega \) and \( \varrho > 0 \) it follows that \( \inf_{S_k} \varrho(x) = \varrho_k > 0 \), that is, the ellipticity is uniform in \( S_k \). Gårding's inequality [3] then implies (i).
From the considerations in section 5 it follows that if \( \varphi \) is any \( \varphi_i \) then the function
\[
\begin{cases}
  l_{\alpha\beta}(\varphi) & \text{for } |x-z| < r, \\
  \varphi & \text{for } |x-z| \geq r,
\end{cases}
\]
is in \( C_0^{[\alpha]+[\beta]}(S) \). This implies
\[
L_{m,m-1}(f,f) = \int \sum_{|\alpha| \leq m, |\beta| \leq m-1} \frac{l_{\alpha\beta}(\varphi)}{\varphi} D_\alpha f \overline{D_\beta f} \, dx
\]
\[
= \int \sum_{|\alpha| \leq m, |\beta| \leq m-1} \frac{l_{\alpha\beta}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \, dx + \int \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} l_{\alpha\beta}^{(1)}(\varphi) D_\alpha f \overline{D_\beta f} \, dx.
\]
Since the function \( l_{\alpha\beta}(\varphi) \) are quadratic forms in the functions (5.5) with coefficients in \( C^{[\alpha]+[\beta]}(\Omega) \) the same is true for \( l_{\alpha\beta}^{(1)}(\varphi) \) (this is easy to check). This implies that
\[
\frac{l_{\alpha\beta}^{(1)}(\varphi)}{\varphi} \in C_0^{[\alpha]+[\beta]}(S).
\]
By iterating the procedure we obtain, writing \( L_{m,m-1} \) for \( L_{m,m-1}(f,f) \),
\[
L_{m,m-1} = \int \sum_{|\alpha| \leq m, |\beta| \leq m-1} \frac{l_{\alpha\beta}^{(0)}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \, dx + \int \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \frac{l_{\alpha\beta}^{(1)}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \, dx + \ldots + \int \sum_{|\alpha| \leq 0, |\beta| \leq m-1} \frac{l_{\alpha\beta}^{(m)}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \, dx,
\]
where we have put \( l_{\alpha\beta}(\varphi) = l_{\alpha\beta}^{(0)}(\varphi) \). From
\[
\left| \frac{l_{\alpha\beta}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \right| \leq \frac{1}{2} r |D_\alpha \varphi f|^2 + r^{-1} \left| \frac{l_{\alpha\beta}(\varphi)}{\varphi} D_\beta f \right|^2
\]
for every \( r > 0 \) it follows that
\[
|L_{m,m-1}| \leq \frac{1}{2} rd_{m-1} D^m \varphi f |^2 + r^{-1} \int \sum_{|\alpha| \leq m, |\beta| \leq m-1} \left| \frac{l_{\alpha\beta}(\varphi)}{\varphi} \right|^2 |D_\beta f|^2 \, dx + \ldots + \int \sum_{|\alpha| \leq 0, |\beta| \leq m-1} \left| \frac{l_{\alpha\beta}(\varphi)}{\varphi} \right|^2 |D_\beta f|^2 \, dx,
\]
that is,
\begin{equation}
|L_{m,m-1}| \leq \frac{1}{2} rd_{m-1} |D^m \varphi f|^2 + r^{-1} \int |\varphi| \leq m-1 \sum_{|\beta| \leq m-1} l_\beta(\varphi) |D^\beta f|^2 \, dx,
\end{equation}

where

\[ l_\beta(\varphi) = \sum_{|\alpha| \leq m} \left| \frac{\partial^{\alpha\beta}(\varphi)}{\varphi} \right|^2 + \sum_{|\alpha| \leq m-1} \left| \frac{\partial^{\alpha\beta}(\varphi)}{\varphi} \right|^2 + \ldots + \sum_{|\alpha| \leq 0} \left| \frac{\partial^{\alpha\beta}(\varphi)}{\varphi} \right|^2. \]

We notice that \( l_\beta(\varphi) \) is a quadratic form in the functions \( (5.5) \) with coefficients in \( C |\beta| (\Omega) \), and that the second term on the right hand side of (6.1) considered as a quadratic form in the derivatives \( D^\beta f \) is of order \( (m-1,m-1) \). We denote it by \( L_{m-1,m-1} \). We have thus

\begin{equation}
|L_{m,m-1}| \leq \frac{1}{2} rd_{m-1} |D^m \varphi f|^2 + r^{-1} |L_{m-1,m-1}|,
\end{equation}

where \( d_k \) as before denotes the number of derivatives of order \( k \). It is now natural to proceed in the same way with \( L_{m-1,m-1} \). By partial integrations \( L_{m-1,m-1} \) is transformed into \( L_{m,m-2} \); all boundary terms disappearing since the coefficients in \( L_{m-1,m-1} \) have compact supports. Then

\[ |L_{m,m-2}| \leq \frac{1}{2} r^2 d_{m-1} |D^m \varphi f|^2 + r^{-2} |L_{m-2,m-2}|, \]

that is,

\begin{equation}
|L_{m-1,m-1}| \leq \frac{1}{2} r^2 d_{m-1} |D^m \varphi f|^2 + r^{-2} |L_{m-2,m-2}|
\end{equation}

for all \( r > 0 \). Analogously

\[ |L_{m-2,m-2}| \leq d_{m-2} |D^{m-1} \varphi f|^2 + |L_{m-3,m-3}|, \]

\[ |L_{m-3,m-3}| \leq d_{m-3} |D^{m-2} \varphi f|^2 + |L_{m-4,m-4}|, \]

\[ \ldots \]

\[ |L_{1,1}| \leq d_1 |D^2 \varphi f|^2 + |L_{0,0}|. \]

By adding these inequalities we obtain

\begin{equation}
|L_{m-2,m-2}| \leq (m-2) d_{m-2} |D^{m-1} \varphi f|^2 + |L_{0,0}|.
\end{equation}

Inserting (6.3) into (6.2) and using (6.4) we obtain after trivial simplifications

\begin{equation}
|L_{m,m-1}| \leq md_m (r |D^m \varphi f|^2 + r^{-3} |D^{m-1} \varphi f|^2) + r^{-3} |L_{0,0}|.
\end{equation}

Here

\[ L_{0,0} = \int m_0(\varphi) f \bar{f} \, dx = \int (m_0(\varphi)/\varphi) \varphi f \bar{f} \, dx, \]

so that

\begin{equation}
|L_{0,0}| \leq \int (s^{-1} |\varphi f|^2 + s |m_0(\varphi)/\varphi|^2 |f|^2) \, dx \leq s^{-1} |\varphi f|^2 + s C_\varphi |f, St \varphi|^2,
\end{equation}

where \( C_\varphi = \max_{\text{St} \varphi} |m_0(\varphi)/\varphi|^2 \) and \( s \) arbitrary \( > 0 \). Taking \( s = tr^3 \) where \( t \) is arbitrary \( > 0 \) we get from (6.5) and (6.6)
\[
(L.7) \quad |L_{m,m-1}| \leq m d_m (r |D^m q f|^2 + r^{-3} |D^{m-1} q f|^2) + r^{-6} t^{-1} |q f|^2 + t C q |f, St q|^2.
\]

According to (iii)
\[
|D^{m-1} q f|^2 \leq r^4 |D^m q f|^2 + \tau (r^4) |q f|^2.
\]

This inequality and (6.7) yield
\[
(L.8) \quad |L_{m,m-1}| \leq 2 m d_m r |D^m q f|^2 + (r^{-3} \tau (r^4) m d_m + r^{-6} t^{-1}) |q f|^2 + t C q |f, St q|^2.
\]

The substitution
\[
r \to (2 m d_m)^{-1} r, \quad t \to C^{-1} t,
\]
in (6.8) gives (ii) and we are finished.

(iii):

This is a classical inequality and is proved by partial integrations and Schwarz's inequality or by a Fourier transformation (e.g. [3]).

(iv):

\[
(q^2 f, f) = (a q_k f, q_k f) + L^{(k)}_{m,m-1} (f, f).
\]

Here \(L^{(k)}_{m,m-1}\) is of the form (ii). By (i)

\[
(6.9) \quad \text{Re} (a q_k f, q_k f) \geq \epsilon_k |D^m q_k f|^2 - \tau_k |q_k f|^2
\]

and according to (ii)

\[
|L^{(k)}_{m,m-1}| \leq \frac{1}{2} \epsilon_k |D^m q_k f|^2 + \tau (t_k) |q_k f|^2 + t_k |f, St q_k|^2
\]

which in combination with (6.9) gives

\[
(6.10) \quad \text{Re} (q^2 f, f) \geq \frac{1}{2} \epsilon_k |D^m q_k f|^2 - \tau (t_k) |q_k f|^2 - t_k |f, St q_k|^2
\]

\[
\geq \frac{1}{2} \epsilon_k |D^m q_k f|^2 + 2 r_k |D^{m-1} q_k f|^2 - \tau'' (t_k, r_k) |q_k f|^2 - t_k |f, St q_k|^2.
\]

Here \(r_k\) and \(t_k\) are arbitrary positive numbers. Now

\[
|D^m q_k f|^2 = |q_k D^m f|^2 + L^{(k)}_{m,m-1},
\]

\[
|D^{m-1} q_k f|^2 = |q_k D^{m-1} f|^2 + L^{(k)}_{m-1,m-2}.
\]

Here \(L^{(k)}_{m,m-1}\) is, of course, not the same form as above but also of type (ii). By (ii)

\[
|L^{(k)}_{m,m-1}| \leq |D^m q_k f|^2 + \tau_0 (t) |q_k f|^2 + t |f, St q_k|^2,
\]

\[
|L^{(k)}_{m-1,m-2}| \leq |D^{m-1} q_k f|^2 + \tau_1 (s) |q_k f|^2 + s |f, St q_k|^2,
\]

where \(t\) and \(s\) are arbitrary \(> 0\), so that

\[
(6.11) \quad 2 |D^m q_k f|^2 \geq |q_k D^m f|^2 - \tau_0 (t) |q_k f|^2 - t |f, St q_k|^2,
\]

\[
(6.12) \quad 2 |D^{m-1} q_k f|^2 \geq |q_k D^{m-1} f|^2 - \tau_1 (s) |q_k f|^2 - s |f, St q_k|^2.
\]
Choose \( t = 8\varepsilon_k^{-1}t_k, \ s = r_k^{-1}t_k. \) Insertion of (6.11) and (6.12) into (6.10) gives

\[
\text{Re}(\varphi_k^2 a f, f) \geq \frac{1}{8} \varepsilon_k |\varphi_k D^m f|^2 + \tau_k |\varphi_k D^{m-1} f|^2 - \tau(r_k, t_k) |\varphi_k f|^2 - 3t_k |f, \text{St} \varphi_k|^2. 
\]

The substitution \( \varepsilon_k \to 8\varepsilon_k, \ 3t_k \to t_k, \) in (6.13) gives (iv).

(v):

Let \( 1 = \sum \varphi_k^2(x) \) be our partition of unity belonging to the covering \((S_k).\)

Set

\[
m_k = \max_{S_v \cap S_k \neq \emptyset} \tau_v.
\]

Define

\[
\tau(x) = \sum m_v \varphi_v^2(x).
\]

For \( x \in S_k \) we have

\[
\tau(x) = \sum_{S_v \cap S_k \neq \emptyset} m_v \varphi_v^2(x) \geq \sum_{S_v \cap S_k \neq \emptyset} \tau_k \varphi_v^2(x) = \tau_k,
\]

that is, \( \inf_{S_k} \tau(x) \geq \tau_k. \) Notice that \( \tau(x) \in C^\infty(\Omega). \)

REFERENCES


UNIVERSITY OF LUND, SWEDEN