VERIFICATION OF A CONJECTURE OF TH. SKOLEM

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In [1], Th. Skolem gives a distribution of the numbers 1, \ldots, 2n into \( n \) disjoint pairs \((a_r, b_r)\) such that \( b_r = a_r + r \) for \( r = 1, 2, \ldots, n \) and \( n \equiv 0 \) or 1 (mod 4). In [2], a set of pairs \((a_r, b_r)\) of this kind was related to a system of disjoint triples \((r, a_r + n, b_r + n)\) over a set of \( 3n \) elements, which, in turn, was used to construct a system of Steiner triples over a set of \( 6n + 1 \) elements. The extension of these methods to the cases \( n \equiv 2 \) or 3 (mod 4) depends on the verification of the following conjecture of Th. Skolem:

**Theorem.** For \( n \equiv 2 \) or 3 (mod 4), the numbers 1, 2, \ldots, 2n−1, 2n+1 can be distributed into \( n \) disjoint pairs \((a_r, b_r)\) such that \( b_r = a_r + r \) for \( r = 1, \ldots, n \).

**Proof.** Let \( n \equiv 2 \) (mod 4). It is sufficient to describe a system of pairs for \( n = 4m + 2 \). Such a system consists of the pairs

A: \((r, 4m + 2 - r)\) for \( r = 1, \ldots, 2m \),
B: \((2m + 1, 6m + 2)\),
C: \((4m + 2, 6m + 3)\),
D: \((4m + 3, 8m + 5)\),
E: \((4m + 3 + r, 8m + 4 - r)\) for \( r = 1, \ldots, m - 1 \),
F: \((5m + 2 + r, 7m + 3 - r)\) for \( r = 1, \ldots, m - 1 \),
G: \((7m + 3, 7m + 4)\).

Let \( X_1 \) be the set of elements which are first elements in the pair \( X \), and \( X_2 \) be the set of second elements. The above pairs are disjoint and cover the set 1, \ldots, 8m+3, 8m+5 because

\[
\begin{align*}
1, \ldots, 2m & \text{ are in } A_1, \\
2m + 1 & \text{ is in } B_1, \\
2m + 2, \ldots, 4m + 1 & \text{ are in } A_2, \\
4m + 2 & \text{ is in } C_1, \\
4m + 3 & \text{ is in } D_1, \\
4m + 4, \ldots, 5m + 2 & \text{ are in } E_1,
\end{align*}
\]

Received December 19, 1960.
$5m + 3, \ldots, 6m + 1$ are in $F_1$,
$6m + 2$ is in $B_2$,
$6m + 3$ is in $C_2$,
$6m + 4, \ldots, 7m + 2$ are in $F_2$,
$7m + 3$ is in $G_1$,
$7m + 4$ is in $G_2$,
$7m + 5, \ldots, 8m + 3$ are in $E_2$,
$8m + 5$ is in $D_2$.

The differences for $r = 1, \ldots, 4m + 2$ are obtained as follows:

$2, 4, \ldots, 4m$ from $A$,
$4m + 2$ from $D$,
$1$ from $G$,
$3, 5, \ldots, 2m - 1$ from $F$,
$2m + 1$ from $C$,
$2m + 3, 2m + 5, \ldots, 4m - 1$ from $E$,
$4m + 1$ from $B$.

This concludes the proof for $n \equiv 2 \pmod{4}$.

Let $n \equiv 3 \pmod{4}$. It is sufficient to describe a system of pairs for $n = 4m - 1$. A system consists of the following pairs

A: $(r, 4m - 1 - r)$ for $r = 1, \ldots, m - 1$,
B: $(m, m + 1)$,
C: $(m + 1 + r, 3m - r)$ for $r = 1, \ldots, m - 2$,
D: $(2m, 4m - 1)$,
E: $(4m, 8m - 1)$,
F: $(4m + r, 8m - 2 - r)$ for $r = 1, \ldots, 2m - 2$,
G: $(2m + 1, 6m - 1)$.

These sets are disjoint and cover the integers $1, \ldots, 8m - 3, 8m - 1$ because

$1, \ldots, m - 1$ are in $A_1$,
$m$ is in $B_1$,
$m + 1$ is in $B_2$,
$m + 2, \ldots, 2m - 1$ are in $C_1$,
$2m$ is in $D_1$,
$2m + 1$ is in $G_1$,
$2m + 2, \ldots, 3m - 1$ are in $C_2$,
$3m, \ldots, 4m - 2$ are in $A_2$,
$4m - 1$ is in $D_2$,
$4m$ is in $E_1$. 
4m + 1, \ldots, 6m - 2 \text{ are in } F_1,
6m - 1 \text{ is in } G_2,
6m, \ldots, 8m - 3 \text{ are in } F_2,
8m - 1 \text{ is in } E_2.

The differences for \( r = 1, \ldots, 4m - 1 \) are obtained as follows:

1 from \( B \),
3, 5, \ldots, 2m - 3 from \( C \),
2m - 1 from \( D \),
2m + 1, 2m + 3, \ldots, 4m - 3 \text{ from } A,
4m - 1 \text{ from } E,
2, 4, \ldots, 4m - 4 \text{ from } F,
4m - 2 \text{ from } G.

This completes the proof of the theorem.

**BIBLIOGRAPHY**


**BOEING AIRPLANE COMPANY, SEATTLE, WASH., U.S.A.**