THE RELATION BETWEEN TWO GENERALISATIONS
OF THE NOTION "SURFACE OF CURVATURE $\leq K$"

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Surfaces of bounded total (Gaussian) curvature $\leq K$ have interesting properties. A. D. Alexandrow has investigated such properties in [2] and especially in [3] and extended them to spaces of curvature $\leq K$ in a more general sense. A region $R$ in a locally compact space of arbitrary dimension with intrinsic metric is called an $R_K$ [3, pp. 36, 41], if for every triangle $T$ in $R$, the sum of the "upper angles" is not greater than the sum of the angles of a triangle $T^K$ with sides of the same lengths on a surface of constant curvature $K$. (If $K > 0$ it is postulated that the perimeter of any triangle $T$ in the $R_K$ is not greater than $2\pi K^{-1}$, so that $T^K$ exists. For the definition of "upper angle" see [3, p. 35] or [2, p. 492].) Then a metric space, in which every point has a neighbourhood which is an $R_K$, is said to be "of curvature $\leq K$" [3, p. 36]. Let us here for the sake of brevity (and to avoid confusion in the sequel) call such a space an $R_K$-space.

In [6] I gave for a surface of total curvature $\leq K$ an estimate for the maximal deviation of a curve $AB$ of given length from the geodesic $AB$. Then I generalised the notion "curvature $\leq K$" in a way different from Alexandrow's—but analogous to Beurling's generalisation for $K = 0$ in [4]—by means of the class $C(K)$ of "functions of curvature $\leq K$". A real-valued function $u(z)$ of a complex variable $z$ is said to belong to $C(K)$ in a region $D$ if it is continuous in $D$ and satisfies

\begin{equation}
L(u, z_0, r) - u(z_0) \geq -\frac{1}{4}K \int_0^r \varrho A(e^{au}, z_0, \varrho) \, d\varrho
\end{equation}

for every $z_0 \in D$ and all sufficiently small $r$ [6, p. 318]. Here $L(u, z_0, r)$ and $A(u, z_0, r)$ denote, as usual in the theory of subharmonic functions, the mean values of $u(z)$ on the circle $|z - z_0| = r$ and the circular disk $|z - z_0| < r$, respectively (cf. [6, p. 317] or Radó [9, p. 3]). Then I called

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defined on $D$, "of curvature $\leq K$" if $u \in C(K)$ in $D$ [6, p. 322].

In this paper we shall see how this notion "surface of curvature $\leq K$" is related to Alexandrow’s notion, here called "$R_K$-space". It will be found that a surface of curvature $\leq K$ in the sense of [6] is also an $R_K$-space. But a 2-dimensional $R_K$-space (or even an $R_K$) need not be a "surface of curvature $\leq K$", because the continuity of $u(z)$ assumed in the definition of the class $C(K)$ is not necessary. However, upper semicontinuity is necessary and sufficient. Furthermore, a general $R_K$-space does not correspond to a region $D$ in the complex $z$-plane but to an arbitrary open Riemann surface $R$. Therefore it is convenient to introduce here a wider class $C'(K)$ of functions $u(z)$ defined on an open Riemann surface $R$:

**Definition.** $u(z) \in C'(K)$ if

a) $u(z)$ is upper semi-continuous,

b) $u(z)$ satisfies (1) for every point $z_0$ of $R$ and all sufficiently small $r$.

Then we have the following two mutually converse theorems:

**Theorem 1.** A surface given by a metric (2), defined on an open Riemann surface $R$ by a real-valued function $u(z) \in C'(K)$, is an $R_K$-space.

**Theorem 2.** Every orientable 2-dimensional $R_K$-space is isometric to an open Riemann surface $R$ with the metric (2), where $u(z) \in C'(K)$.

**Proof of Theorem 1.** We shall first show that any $u(z)$ in $C'(K)$ can locally be represented as a logarithmic potential. Let $D$ be a region of $R$, which may be considered as situated in a $z$-plane. Let us introduce in $D$ the auxiliary function

$$v(z) = -(2\pi)^{-1} \int_D \ln |z - \zeta| \ K e^{2u(z)} \ d\xi d\eta,$$

where we have put $\zeta = \xi + i\eta$. Then the function $u - v$ is upper semi-continuous, because $u$ is so and $v$ is continuous. And for a sufficiently small circle in $D$ we find

$$L(u-v, z_0, r) - u(z_0) + v(z_0) = L(u, z_0, r) - u(z_0) - [L(v, z_0, r) - v(z_0)] \geq 0 ,$$

because $u$ satisfies (1) and a calculation of $L(v, z_0, r) - v(z_0)$ yields exactly the right member of (1). Indeed, we have

$$\int_0^{2\pi} \ln |z_0 - \zeta + re^{i\theta}| \ d\theta = \begin{cases} 2\pi \ln |z_0 - \zeta| & \text{for } |z_0 - \zeta| > r , \\ 2\pi \ln r & \text{for } |z_0 - \zeta| \leq r , \end{cases}$$
which may be obtained by applying Jensen's formula (cf. e.g. [1, p. 185]) to the analytic function \( f(z) = z - (\zeta - z_0) \), and thus

\[
L(v, z_0, r) - v(z_0) = -(2\pi)^{-2}K \int_0^{2\pi} \left[ \int_D \left( \ln |z_0 + re^{i\theta} - \zeta| - \ln |z_0 - \zeta| \right) e^{2u(z)} \, d\xi \, d\eta \right] d\theta
\]

\[
= -(2\pi)^{-1}K \int_0^{2\pi} \int_0^r \int_0^{2\pi} \left( \ln r - \ln \sigma \right) \exp 2u(z_0 + \sigma e^{i\varphi}) \, d\sigma \, d\varphi
\]

\[
= -(2\pi)^{-1}K \int_0^{2\pi} \int_0^r \int_0^{2\pi} \sigma^{-1} d\varphi \exp 2u(z_0 + \sigma e^{i\varphi}) \, d\sigma \, d\varphi
\]

\[
= -(2\pi)^{-1}K \int_0^{2\pi} \int_0^r \left[ \sigma^{-1} \int_0^{2\pi} \exp 2u(z_0 + \sigma e^{i\varphi}) \, d\varphi \right] d\varphi
\]

\[
= -\frac{1}{2}K \int_0^r \sigma A(e^{2u}, z_0, \sigma) \, d\varphi.
\]

Thus \( u - v \) is subharmonic in \( D \). In a region \( D' \), which together with its boundary is contained in \( D \), \( u - v \) is then the potential of a non-positive mass-distribution \(-\mu\) plus a harmonic function [9, p. 42]:

\[
(3) \quad u(z) - v(z) = \int_{D'} \ln |z - \zeta| \, \mu(dE_{\zeta}) + h(z).
\]

On the other hand, if we divide the integral over \( D \) defining \( v \) into two parts, one over \( D' \) and the other over \( D - D' \), the latter part gives in \( D' \) a harmonic function, and we get in \( D' \)

\[
(4) \quad v(z) = -(2\pi)^{-1} \int_{D'} \ln |z - \zeta| \, K e^{2u(z)} \, d\xi \, d\eta + h_1(z)
\]

\[
= -(2\pi)^{-1}K \int_{D'} \ln |z - \zeta| \, j(dE_{\zeta}) + h_1(z),
\]

where \( j(E) \) denotes the area (Lebesgue measure) of the set \( E \) in the metric (2). Adding (3) and (4) we find

\[
u(z) = -(2\pi)^{-1} \int_{D'} \ln |z - \zeta| \, \omega(dE_{\zeta}) + h_2(z)
\]

with \( h_2(z) \) harmonic in \( D' \) and

\[
(5) \quad \omega(E) = K j(E) - 2\pi \mu(E).
\]
Now according to a theorem of Reschetnjak [10] (cf. [2, pp. 503 ff.]), the metric (2), with \( u(z) \) a difference between two subharmonic functions, is "of bounded curvature" in the sense of [2, p. 493], and the curvature corresponding to a set \( E \) is \( \omega(E) \). As \( \mu(E) \geq 0 \), we get from (5)

\[
\omega(E) \leq K j(E)
\]

for every Borel set \( E \) in \( D' \). However, (6) then holds also for any Borel set \( E \) in \( D \). Indeed such an \( E \) can be approximated from within by a closed \( E' \) for which the differences \( |\omega(E) - \omega(E')| \) and \( |j(E) - j(E')| \) are arbitrarily small. If \( \omega(E) - K j(E) = 2\varepsilon > 0 \), we could thus find an \( E' \) in a \( D' \) so that

\[
\omega(E') > K j(E) + \varepsilon > K j(E')
\]

which is impossible since (6) has been proved for \( E' \). Thus (6) must hold for any Borel set in \( D \). An arbitrary region \( E \) in \( R \) can be divided into parts \( E_i \) for which this yields \( \omega(E_i) \leq K j(E_i) \). Adding, we get (6) for \( E \) or

\[
\omega(E)/j(E) \leq K,
\]

that is: The metric is of specific curvature \( \leq K \) in the sense of [2, p. 431]. According to Alexandrow [2, p. 513], the surface is then an \( R_K \)-space. (In [2] this statement is proved only for convex surfaces (cf. theorem 4, p. 433, and its proof on pp. 442f. in [2]). For that part of theorem 4 with which we are concerned the method of this proof is however general.)

Proof of theorem 2. According to Alexandrow [3, p. 43], an \( R_K \)-space is also "of bounded curvature". Huber has recently proved [7, p. 100] (completing a result of Reschetnjak [10]) that an orientable 2-dimensional space "of bounded curvature" is isometric to a Riemann surface \( R \) with the metric

\[
ds = e^{u(z)} \lvert dz \rvert,
\]

where \( u(z) \) is the difference between two subharmonic functions. \( R \) is open, because Alexandrow's definition of \( R_K \)-space (mentioned above) assumes that every point in the space has a neighbourhood. We have to prove that \( u(z) \) a) satisfies (1) and b) is upper semi-continuous.

a) (1) is trivial, if \( u(z_0) = -\infty \). Further \( u(z_0) = +\infty \) is impossible in an \( R_K \), as will follow from b). Here we may thus assume \( u(z_0) \) finite. Then we prove (1) for \( r < \frac{1}{2} \) so small that the closed disk \( \lvert z - z_0 \rvert \leq r \) is contained in \( R \). Let \( C_r \) denote the interior of such a disk. Then \( u(z) \) may be written (cf. formula (3) in [2, p. 504])

\[
u(z) = - (2\pi)^{-1} \iint_{C_r} \ln \lvert z - \zeta \rvert \omega(dE_\zeta) + h(z),
\]
where \( h(z) \) is a function harmonic in \( C_q \), and \( \omega(E) \) denotes the curvature of the set in our \( R-K \)-space which corresponds to \( E \) in \( R \) (The curvature of a Borel set being defined in [2, p. 496]).

As \( L(h, z_0, r) - h(z_0) = 0 \), it is sufficient to study the left member of (1) for

\[
u_1(z) = u(z) - h(z) = -(2\pi)^{-1} \int \ln |z - \zeta| \omega(dE_\zeta) .
\]

Inserting this expression in \( L(u_1, z_0, r) \), we obtain

\[
L(u_1, z_0, r) = (2\pi)^{-1} \int_0^{2\pi} u_1(z_0 + re^{iv}) dv = -(2\pi)^{-2} \int_0^{2\pi} \left[ \int \ln |z_0 + re^{iv} - \zeta| \omega(dE_\zeta) \right] dv.
\]

Here we interchange the order of integration. This is legitimate according to well-known theorems of Tonelli and Fubini (cf. e.g. [8, p. 151]), because the integrand is measurable and \( r < 0 \) in \( C_r \) (in virtue of the assumption \( r < \frac{1}{2} \) and the integral with respect to the measure \( |\omega| \) is finite. This last point is verified at the end of our calculation. Thus we get

\[
L(u_1, z_0, r) = -(2\pi)^{-2} \int \left[ \int \ln |z_0 + re^{iv} - \zeta| dv \right] \omega(dE_\zeta).
\]

The integral in brackets has (as mentioned above p. 340) the value \( 2\pi \ln r \), since \( |\zeta - z_0| < r \) in \( C_r \), and we get

\[
L(u_1, z_0, r) = -(2\pi)^{-1} \ln r \int \omega(dE_\zeta) = -(2\pi)^{-1} \ln r \omega(C_r)
\]

(cf. the analogous result 4.29 in [9, p. 30]). Now we see that the corresponding integral with respect to \( |\omega| \) has the value \( -(2\pi)^{-1} \ln r |\omega|(C_r) \).

That \( |\omega|(C_r) \) is finite is contained in the statement that the space is "of bounded curvature" [2, p. 493].

Now, when \( u_1(z_0) \) is finite, the left member of (1) is,

\[
L(u_1, z_0, r) - u_1(z_0) = -(2\pi)^{-1} \ln r \int \omega(dE_\zeta) + (2\pi)^{-1} \int \ln |z_0 - \zeta| \omega(dE_\zeta)
\]

\[
= -(2\pi)^{-1} \int \ln \frac{r}{|z_0 - \zeta|} \omega(dE_\zeta).
\]

However, any \( R-K \)-space is also of specific curvature \( \leq K \), that is

\[
\omega(E) \leq K j(E)
\]
for every region \( E \). (This fact is stated by Alexandrow in [2, p. 513]. A proof is given here in an appendix.) Then (6) must hold for every Borel set \( E \) in \( R \). Applying it to the last member of (8), we get,

\[
(9) \quad L(u, z_0, r) - u(z_0) = L(u_1, z_0, r) - u_1(z_0) \geq - (2\pi)^{-1} K \int_{C_r} \ln \frac{r}{|z_0 - \zeta|} j(dE_\zeta),
\]

because \( \ln (r/|z_0 - \zeta|) > 0 \) in \( C_r \).

By the definition of \( A(u, z_0, r) \) the right member of (1) is

\[
- \frac{1}{2} K \int_0^r \left( \frac{1}{\pi q^2} \int_{C_{z_0}} e^{2 \pi u(x, y)} d\xi d\eta \right) dq = - (2\pi)^{-1} K \int_0^r \int_{C_{z_0}} \ln \frac{r}{|z_0 - \zeta|} j(dE_\zeta) dq.
\]

This is a triple integral over the cone \( |\zeta - z_0| < q, 0 < q < r \), which may be written

\[
- (2\pi)^{-1} K \int_{C_r} \int_{|z_0 - \zeta|} q^{-1} dq \int_{C_{z_0}} j(dE_\zeta) = - (2\pi)^{-1} K \int_{C_r} \ln \frac{r}{|z_0 - \zeta|} j(dE_\zeta).
\]

This is the right member of (9), and thus, (1) is proved.

b) We start the proof that \( u(z) \) is upper semi-continuous by splitting the curvature \( \omega \) into its positive and negative parts: \( \omega = \omega^+ + \omega^- \). The corresponding parts of the potential \( u \) are denoted by \( u^+ \) and \( u^- \) resp. Then \( u^- \) is a subharmonic function and thus upper semi-continuous. It remains to prove the upper semi-continuity of \( u^+ \). In fact we can prove much more, namely that \( u^+(x + iy) \) has partial derivatives \( u^+_x \) and \( u^+_y \), which satisfy Lipschitz conditions of every order \( 1 - \varepsilon < 1 \). Since \( u^+ \equiv 0 \) if \( K \leq 0 \), we may assume \( K > 0 \) in the sequel.

In a circular disk \( C \) in \( R \) we have

\[
(10) \quad u^+(z) = - (2\pi)^{-1} \int_{C} \ln |z - \zeta| \omega^+(dE_\zeta) + h(z),
\]

where \( h(z) \) is harmonic in \( C \). Introducing a positive number \( \varepsilon < 1 \), we can also write

\[
u^+(z) = \frac{\omega^+(C)}{2\pi \varepsilon} \int_{C} \ln |z - \zeta|^{-\varepsilon} \frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z).
\]

The integral here is the logarithm of a geometric mean. The inequality between the arithmetic and geometric means then gives

\[
u^+(z) \leq \frac{\omega^+(C)}{2\pi \varepsilon} \ln \int_{C} |z - \zeta|^{-\varepsilon} \frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z).
\]

For a given \( \varepsilon > 0 \) we can choose \( C \) with given centre \( z_0 \) so that \( \omega^+(C) \leq \pi \varepsilon \). Indeed, when the radius \( a \) of \( C \) tends to 0, \( \omega^+(C) \) tends to \( \omega^+(z_0) = 0 \).
In an \( R_K \), \( \omega^+(z_0) > 0 \) is impossible. (This follows e.g. from the fact that no shortest line ("Kürzeste") could pass through such a point. However, according to theorem 6, p. 54, in [3], in an \( R_K \) a shortest line varies continuously with its endpoints. Hence, between two shortest lines, \( AB \) and \( AC \), passing near \( z_0 \) on opposite sides, there must be an intermediate one passing through \( z_0 \).) Thus, if \( a < \frac{1}{2} \), we have

\[
u(z) \leq \frac{1}{2} \ln \int_C |z - \zeta|^{-\varepsilon} \frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z) + u^-(z)\]

And because \( h(z) \) and \( u^-(z) \) have upper bounds in \( C \), we get in \( C \)

\[
e^{2\nu(z)} \leq k \int_C |z - \zeta|^{-\varepsilon} \omega^+(dE_\zeta),
\]

where \( k \) is a constant.

We can now estimate \( \omega^+(\gamma_r) \) for an arbitrary circular disk \( \gamma_r \) in \( C \) with radius \( r \). Using (6), the last inequality for \( e^{2\nu} \) and that \( K > 0 \) has been assumed, we get

\[
\omega^+(\gamma_r) \leq K \int_{\gamma_r} e^{2\nu(z)} \, dx \, dy
\]

\[
\leq K \int_{\gamma_r} \left( k \int_C |z - \zeta|^{-\varepsilon} \omega^+(dE_\zeta) \right) \, dx \, dy
\]

\[
= Kk \int_C \left( \int_{\gamma_r} |z - \zeta|^{-\varepsilon} \, dx \, dy \right) \omega^+(dE_\zeta)
\]

\[
\leq Kk \int_C \left( 2\pi \rho^{1-\varepsilon} \, d\rho \right) \omega^+(dE_\zeta)
\]

\[
= \frac{2\pi}{2 - \varepsilon} K kr^{2-\varepsilon} \int_C \omega^+(dE_\zeta) = k'r^{2-\varepsilon}.
\]

We have thus proved that, for every point \( z_0 \) in \( R \) and every \( \varepsilon > 0 \), there exists a circle \( C \) with centre \( z_0 \) such that

\[
(11) \quad \omega^+(\gamma_r) \leq k'r^{2-\varepsilon}
\]

for every circle \( \gamma_r \) in \( C \).

Studying the regularity of \( u^+(z) \) at a point \( z_0 \), we need only the values of \( u^+(z) \) in a corresponding circle \( C \). In the expression (10) for \( u^+(z) \) we may also disregard the function \( h(z) \), which has partial derivatives of the second order, and the constant \(-\frac{(2\pi)^{-1}}{2-\varepsilon}\). Instead of \( u^+(z) \) we thus study
\[ t(z) = \iint_{C} \ln |z - \zeta| \omega^{+}(dE_{\zeta}) = \frac{1}{2} \iint_{C} \ln [(x - \xi)^{2} + (y - \eta)^{2}] \omega^{+}(dE_{\xi+\eta}) . \]

For the derivatives of \( t(x, y) \) we get

\[ t_{x} = \iint_{C} \frac{x - \xi}{(x - \xi)^{2} + (y - \eta)^{2}} \omega^{+}(dE_{\xi+\eta}) \]

and an analogous expression for \( t_{y} \), or in one formula

\[ t_{1} = t_{x} - it_{y} = \iint_{C} \frac{\bar{z} - \bar{\xi}}{|z - \zeta|^{2}} \omega^{+}(dE_{\zeta}) = \iint_{C} \frac{\omega^{+}(dE_{\zeta})}{z - \zeta} . \]

The differentiation under the integral sign may be justified by inverting the order of integration in

\[ \iint_{C} \left( \int_{z_1}^{z_2} \frac{x - \xi}{(x - \xi)^{2} + (y - \eta)^{2}} \, dx \right) \omega^{+}(dE_{\zeta}) . \]

We put \(|z - z_0| = r\), suppose \( 4r < a \) (radius of \( C \)), and denote by \( C' \) the circular disk with centre \( z_0 \) and radius \( 2r \). Now we can estimate \( t_{1}(z) - t_{1}(z_0) \) in the following way:

\[ t_{1}(z) - t_{1}(z_0) = \iint_{C} \left( \frac{1}{z - \zeta} - \frac{1}{z_0 - \zeta} \right) \omega^{+}(dE_{\zeta}) \]

\[ = \iint_{C} \frac{1}{z - \zeta} \omega^{+}(dE_{\zeta}) - \iint_{C} \frac{1}{z_0 - \zeta} \omega^{+}(dE_{\zeta}) + \iint_{C-C'} \frac{z_0 - z}{(z - \zeta)(z_0 - \zeta)} \omega^{+}(dE_{\zeta}) \]

\[ = I_1 + I_2 + I_3 . \]

For \( I_2 \) we get the estimate

\[ |I_2| \leq \iint_{C'} \frac{\omega^{+}(dE_{\zeta})}{|z_0 - \zeta|} . \]

The most unfavourable mass-distribution compatible with (11) is

(12) \[ \omega^{+}(C_{q}) = k' q^{2-\varepsilon} \]

for every circle \( C_{q} \) with centre \( z_0 \) and radius \( q \). This gives

\[ |I_3| \leq \int_{0}^{2r} k'(2 - \varepsilon) q^{1-\varepsilon} q^{-1} dq = \frac{k'(2 - \varepsilon)}{1 - \varepsilon} (2r)^{1-\varepsilon} = c_2 r^{1-\varepsilon} . \]

For \( I_1 \) we have an analogous estimate \(|I_1| \leq c_1 r^{1-\varepsilon} \). For \( I_3 \) we find
\[ |I_3| = r \left| \int_{C-C'} \int \frac{\omega^+(dE_\zeta)}{(z-\zeta)(z_0-\zeta)} \right| \leq 2r \int_{C-C'} \frac{\omega^+(dE_\zeta)}{|z_0-\zeta|^2}. \]

In this case the most unfavourable mass-distribution is given by (12) for \( \rho > 2r \) and has the mass \( k'(2r)^{2-\varepsilon} \) concentrated on the circle \( |\zeta - z_0| = 2r \). This yields
\[ |I_3| \leq 2r \left[ k'(2r)^{2-\varepsilon}(2r)^{-2} + \int_{2r}^{a} k'(2-\varepsilon)q^{1-\varepsilon}q^{-2} dq \right] \]
\[ = k'(2r)^{1-\varepsilon} + 2r \frac{k'(2-\varepsilon)}{-\varepsilon} [a^{-\varepsilon} - (2r)^{-\varepsilon}] \leq c_3 r^{1-\varepsilon}. \]

With these estimates of \( I_1, I_2 \) and \( I_3 \) we get
\[ |t_1(z) - t_1(z_0)| \leq |I_1| + |I_2| + |I_3| \leq (c_1 + c_2 + c_3) r^{1-\varepsilon}, \]
which is the desired Lipschitz condition (\( \varepsilon > 0 \) can be made arbitrarily small). (The estimation of \( t_1(z) - t_1(z_0) \) can be carried out in a more elegant way by the method used by Carleson in [5, pp. 17–18 (II)].)

Remark 1. The result of b) also shows, that the superharmonic part \( u^+(z) \) of any function \( u(z) \in C'(K) \) has partial derivatives \( u^+_x \) and \( u^+_y \), which satisfy Lipschitz conditions of every order \( 1 - \varepsilon < 1 \).

Remark 2. The estimate—mentioned in the introduction—for the deviation of a curve \( AB \) from the geodesic \( AB \) has been proved for an arbitrary \( R_K \) by Alexandrow [3, p. 82]. For curves \( \gamma \) of given length \( l \) in an \( R_K \), connecting endpoints with given geodesic distance \( r \), the deviation is greatest when the \( R_K \) is of constant curvature \( K \) and \( \gamma \) consists of two geodesics of equal length \( \frac{1}{2} l \). In my previous paper [6], I was not able to generalise this estimate to a metric (2) with \( u \in C(K) \) for \( K > 0 \) (cf. theorem 1, p. 317, and theorem 3, p. 326, in [6]). Theorem 1 above shows that this case, too, is contained in Alexandrow’s result.

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Appendix: On the notion of area in a 2-dimensional \( R_K \). \(^1\)

On p. 343 we used the fact, stated by A. D. Alexandrow in [2, p. 513], that a 2-dimensional \( R_K \)-space is of specific curvature \( \leq K \). Since Alexandrow’s proof has not yet been published, a proof is given here with his

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consent. This proof is intimately connected with the notion of area in an $R_K$. Indeed, if $\omega$ and $j$ as above denote curvature and area, respectively, we have to prove

\[(\omega(E) \leq K j(E)) \tag{6}\]

for any region $E$. According to the definition of an $R_K$, we know that for any triangle $T$ the “excess relative $K$” $\delta_K(T)$ is $\leq 0$. In the simplest case this is equivalent to $\omega(T) \leq K j(T^K)$, where $T^K$, as above (p. 339) and everywhere in the following, denotes a triangle in a “$K$-plane” (surface of constant curvature $K$) [3, p. 34] with sides of the same lengths as the sides of $T$. According to theorem 1, p. 71 in [3], $j(T^K) \geq j(T)$. Then for $K \leq 0$, the inequality (6) follows for certain triangles. But for $K > 0$ we need an estimate of the difference $j(T^K) - j(T)$ corresponding to the theorem on p. 399 in [2]. Therefore we must carry out a discussion corresponding to § 1 of chap. X in [2, pp. 391 ff.]. This will also lead to the conclusion that the notion of area defined in [3, pp. 70 ff.], for a 2-dimensional $R_K$ may be understood also in the stronger sense of [2, chap. X].

We shall need the following elementary estimate of the area of a triangle in a $K$-plane in terms of one angle and the greatest side.

**Lemma 1.** The area of a triangle in a $K$-plane with greatest side $d < |K|^{-1}$ and one angle $\nu$ is less than $\nu d^2$.

**Proof.** The area of the triangle is at most $\nu/2\pi$ times the area of a circle with radius $d$. The area of this circle is for $K > 0$:

$$2\pi K^{-1}[1 - \cos(dK^\dagger)] < \pi d^2$$

for $K < 0$:

$$2\pi K^{-1}[1 - \cosh(d|K|^\dagger)] = \pi d^2 \cosh(\theta d|K|^\dagger) < \pi d^2 \cosh 1$$

(and for $K = 0$: $\pi d^2$). In all cases we get even better estimates than stated in the lemma.

Now we begin the investigation corresponding to Alexandrow’s. The curvature $\omega$ has here to be replaced by the excess relative $K$, that is

$$\delta_K(T) = \alpha + \beta + \gamma - \alpha K - \beta K - \gamma K,$$

where $\alpha$, $\beta$, $\gamma$ denote the angles of $T$ and $\alpha K$, $\beta K$, $\gamma K$ are the angles of the “corresponding triangle” $T^K$ in a $K$-plane. Instead of the lemma of [2, p. 392] on triangles with polyedric metric, we have a lemma on triangles with “concavely $K$-polyedric” metric. We call an intrinsic metric of a 2-dimensional manifold $S$ concavely $K$-polyedric, if every point of $S$ has a neighbourhood, which is isometric to a cone in a space
of constant curvature $K$, and the “full angle” [2, p. 38] of any point of $S$ is $\geq 2\pi$.

**Lemma 2.** If $T$ is a triangle on a 2-dimensional manifold with concavely $K$-polyedric metric, $j(T)$ and $j(T^K)$ are the areas of $T$ and the corresponding triangle $T^K$, respectively, $d < |K|^{-\frac{1}{2}}$ is the diameter of $T$ and $\delta_K(T)$ is the relative excess of $T$, then

$$\delta_K(T)d^2 \leq j(T) - j(T^K) \leq 0.$$ (13)

**Proof.** The right inequality is well known—even for any $R_K$ (Theorem 1, p. 71, in [3]). The problem is to prove the left inequality.

The difference between $T$ and $T^K$ is due to the presence of “conic points” in the interior and on the sides of $T$. For the following proof it is important to observe that no interior point has a full angle $\geq 3\pi$. In fact, if $T$ (with vertices $A, B, C$) should contain a point $P$ with full angle $\geq 3\pi$, $\not\angle APB$ or one of the analogous angles would have to be $\geq \pi$. This is impossible.

Now any interior conic point $P$ can be removed or displaced to the boundary of $T$ by the following construction, (cf. fig. 1 and [2, pp. 394f.]).

![Fig. 1.](image)

We connect $P$ with one vertex $A$ of $T$ by the geodesic $AP$, and suppose that no other conic point is situated on $AP$. (Otherwise we begin with the first conic point on $AP$.) Then we draw a geodesic from $P$ so that the angles which it forms with $PA$ are equal, say $\varphi$. We have $\pi < \varphi < \frac{3}{2}\pi$. This geodesic is extended until it in $Q$ meets either the side $BC$ or a conic point. We make a cut along $APQ$ and insert between its sides two triangles $AP'Q$ and $AP''Q$ from a $K$-plane with the sides $AP' = AP'' = AP$ and $P'Q = P''Q = PQ$ and the angles at $P'$ and $P''$ equal to $2\pi - \varphi$. If then the sides $AQ$ of the two triangles are identified, $T$ is
transformed into a new triangle $T'$. The excess — and because $T^K$ is unchanged, also the excess relative $K$ — of the triangle has increased as much as the angle at $A$, say by $2v$. The area of the triangle has increased by the two congruent triangles $AP'Q$ and $AP''Q$. According to lemma 1, either of these triangles has an area less than $vd^2$. Thus, if (13) should not be true for $T$, we would have

\[ j(T') - j(T^K) < j(T) - j(T^K) + 2vd^2 < \delta_K(T)d^2 + 2vd^2 = \delta_K(T'')d^2, \]

that is, (13) could not hold for $T'$ either. (The diameter of $T'$ is also $d$, because for any not too large triangle in an $R_K$ the diameter is the greatest side. This follows from the corresponding fact for the $K$-plane by theorem 2, p. 53, in [3].) The number of conic points in the interior of $T'$ is less than in $T$. Repeating this process we can remove all conic points from the interior of $T$. Thus it will suffice to prove (13) for triangles $T$ without conic points in the interior.

Such a triangle $T$ may differ from $T^K$ by the presence of extra vertices $P$, with angles $> \pi$, on the sides. Now any such vertex $P$ can be removed by the following transformation of $T$ into a new triangle $T_1$ (cf. fig. 2, and consider e.g. the case $K=0$ of the ordinary plane). If the vertices on $AB$ are $A, P, D, \ldots$ in that order, we first extend $DP$ to $A'$ so that $PA' = PA$ (and the angle $DPA' = \pi$). Then we construct in a $K$-plane the triangle $A_1B_1C_1$ with sides equal to $A'B$, $BC$ and $CA$. In this triangle the polygons $A_1P_1D_1 \ldots B_1$, $B_1E_1 \ldots C_1$ and $C_1F_1 \ldots A_1$ are constructed congruent to $A'PD \ldots B$, $BE \ldots C$ and $CF \ldots A$, respectively. $A_1D_1 \ldots B_1E_1 \ldots C_1F_1 \ldots A_1$ is our new triangle $T_1$, where the extra vertex $P$ is removed but all other extra vertices remain with unchanged angles. Using arguments similar to the proof of lemma 2 in
[3, pp. 51f.], we find that each angle of $T_1$ is greater than the corresponding angle of $T$. In fact

$$\not\angle BPA' > \not\angle BPA \Rightarrow B_1A_1 = BA > BA$$
$$\Rightarrow \not\angle B_1C_1A_1 > \not\angle BCA \Rightarrow \not\angle C_1 > \not\angle C,$$

$$\not\angle APC > \not\angle A'PC \Rightarrow A_1C_1 = AC > A'C$$
$$\Rightarrow \not\angle A_1B_1C_1 > \not\angle A'BC \Rightarrow \not\angle B_1 > \not\angle B,$$

$$\not\angle P_1B_1C_1 > \not\angle PBC \Rightarrow P_1C_1 > PC$$
$$\Rightarrow \not\angle P_1A_1C_1 > \not\angle PAC \Rightarrow \not\angle A_1 > \not\angle A.$$

For comparison of the areas of $T_1$ and $T$ we also construct in $T_1$ the triangles $A_1C_1P'$ and $B_1C_1P''$ congruent to $ACP$ and $BCP$, respectively. The difference $\Delta j$ between the areas $j(T_1)$ and $j(T)$ is, in fact, equal to the difference between the sums $j(A_1C_1P_1) + j(B_1C_1P_1)$ and $j(ACP) + j(BCP) = j(A_1C_1P') + j(B_1C_1P'')$, because these sums differ from $j(T_1)$ and $j(T)$, respectively, by areas of polygons which are congruent in pairs. $\Delta j$ is thus equal to the sum of the areas of the triangles $A_1P_1P'$, $C_1P_1P'$, $C_1P_1P''$ and $B_1P_1P''$. Estimating these areas by the formula $j < \psi d^2$ of lemma 1, we find

$$\Delta j < \psi d^2,$$

where $\psi = \not\angle P_1A_1P' + \not\angle P'C_1P'' + \not\angle P_1B_1P''$ is equal to the increase in excess $\delta(T_1) - \delta(T) = \delta_K(T_1) - \delta_K(T)$. Thus if (13) should not be true for $T$, we would have

$$j(T_1) - j(T^K) < j(T) - j(T^K) + \psi d^2 < \delta_K(T) d^2 + \psi d^2 = \delta_K(T_1) d^2,$$

that is, (13) could not hold for $T_1$ either.

Repeating this process we can successively remove all extra vertices from $T$. We thus arrive at the result that if (13) did not hold for $T$, it could not hold for $T^K$ either. But as (13) is obviously true for $T^K$, it must be true also for $T$, and thus the proof is completed.

Corresponding to the lemma on p. 396 in [2] we have the following result:

**Lemma 3.** Let $T$ be a triangle in a 2-dimensional $R_K$, $d < |K|^{-1}$ its diameter, $\delta_K(T)$ its excess relative $K$ and $j(T^K)$ the area of the corresponding triangle in the $K$-plane. Then for any partition of $T$ into triangles $T_i$, the sum of the areas $j(T_i^K)$ of the corresponding triangles in the $K$-plane satisfies the inequality

$$\delta_K(T)d^2 \leq \sum_i j(T_i^K) - j(T^K) \leq 0.$$

**Proof.** Let $T$ be divided into triangles $T_i$. If each $T_i$ is replaced by the corresponding triangle $T_i^K$ in a $K$-plane, the $T_i^K$ (connected in the
same way as the $T_i$) constitute a polygon $Q$ with $K$-polyedric metric.
Any angle $\alpha_i$ in a $T_i$ is not greater than the corresponding angle $\alpha_i^K$ in
the $T_i^K$ (theorem 4, p. 54, in [3]). This has three consequences:

1° For any interior vertex of $Q$ the full angle is $\geq 2\pi$. In fact, it is
not smaller than the full angle of the corresponding point in $T$, and even
this is $\geq 2\pi$.

2° At any exterior vertex of $Q$ which does not correspond to one of
the three vertices of $T$ the angle is $\geq \pi$. This angle is not smaller than
the corresponding angle in $T$, which also is $\geq \pi$. (The “Schwenkung”
[2, pp. 351ff.] of a geodesic is non-positive [2, p. 498].) Thus, considering
its interior metric $Q$ is a triangle.

3° The relative excess of $Q$ is not smaller than the relative excess of $T$:
$$\delta_K(Q) \geq \delta_K(T).$$

Because of 1° and 2°, $Q$ satisfies the assumptions of lemma 2, which
yields
$$0 \geq j(Q) - j(T^K) \geq \delta_K(Q)d^2 \geq \delta_K(T)d^2,$$
because of 3° and since the diameter of $Q$ is equal to the greatest side of
$Q=$the greatest side of $T=d$. However, (14) is the statement of our
lemma, because $j(Q) = \Sigma_i j(T_i^K)$.

**Theorem.** Every triangle $T$ in a 2-dimensional $R_K$ has an area $j(T)$ in
the following sense: Let $T$ be divided into triangles $T_i$ with diameters
$\leq d < |K|^{-1}$. Then if $d$ tends to zero, the sum $\Sigma_i j(T_i^K)$ of the areas of the
triangles $T_i^K$ (in a $K$-plane and with sides of the same lengths as the sides
of $T_i$) converges to the limit $j(T)$. More precisely the inequality
$$0 \leq \Sigma_i j(T_i^K) - j(T) \leq -\delta_x(T)d^2$$
holds, where $\delta_x(T)$ is the excess relative $x$ and $x = \max(K, 0)$.

**Proof.** We first consider two arbitrary partitions of $T$ into triangles
$T_i$ with diameters $\leq d$ and into triangles $T_h'$ with diameters $\leq d_1 < |K|^{-1}$,
respectively, and prove that
$$\delta_x(T)d_1^2 \leq \Sigma_i j(T_i^K) - \Sigma_h j(T_h'^K) \leq -\delta_x(T)d^2.$$ 
This is done by means of a common subdivision of the two partitions,
that is, a partition of $T$ into triangles $T''_v$ so that any $T''_v$ is at the same
time contained in one $T_i$ and one $T_h'$. According to lemma 3 we have
for any $T_i$
$$\delta_K(T_i)d^2 \leq \Sigma'_v j(T''_v^K) - j(T_i^K) \leq 0,$$ 
where the sum $\Sigma'$ is extended over those $v$ for which $T''_v$ is contained
in $T_i$. Adding for all $T_i$ we find
(16) \[ d^2 \sum_i \delta_K(T_i) \leq \sum_r j(T_r^K) - \sum_i j(T_i^K) \leq 0. \]

Between the excesses of $T$ and the $T_i$ we have the relation (obtained by considering the sum of all the angles of the $T_i$)

(17) \[ \sum_i \delta(T_i) = \delta(T) - \sum_P \tau_P - \sum_Q \omega_Q \geq \delta(T), \]

because both the "Schwenkung" $\tau_P$ of one side of $T$ at a point $P$ (vertex of some $T_i$) and the curvature $\omega_Q$ of an interior vertex $Q$ are $\leq 0$. Thus we have

\[ \sum_i \delta_K(T_i) = \sum_i [\delta(T_i) - Kj(T_i^K)] \geq \delta(T) - K \sum_i j(T_i^K). \]

For $K \geq 0$ we now use the fact that $\sum_i j(T_i^K) \leq j(T^K)$. This is the right inequality of lemma 3 but does not depend on the assumption made there about the diameter of $T$. In fact, it follows directly from theorem 1 in [3, p. 71] (cf. the proof of lemma 3). We thus find

(18) \[ \sum_i \delta_K(T_i) \leq \delta(T) - Kj(T^K) = \delta_K(T) = \delta_x(T). \]

If $K < 0$ we can replace $\delta_K$ by $\delta$. We find by (17)

(19) \[ \sum_i \delta_K(T_i) = \sum_i [\delta(T_i) - Kj(T_i^K)] \geq \sum \delta(T_i) \geq \delta(T) = \delta_x(T). \]

Inserting (18) and (19) in the left member of (16) we get

\[ d^2 \delta_x(T) \leq \sum_r j(T_r''^K) - \sum_i j(T_i^K). \]

Using

\[ \sum_r j(T_r''^K) \leq \sum_h j(T_h^K), \]

which is the right inequality (16) for the partition into the triangles $T_h'$, we get the right inequality (15). The left inequality is proved in the same way.

For a sequence of partitions $P_n$ of $T$ into triangles $T_v^{(n)}$ with diameters $\leq d_n$, (15) yields, if we put $\sum n j(T_v^{(n)K}) = \Sigma n$

\[ \delta_x(T)d_n^2 \leq \Sigma n - \Sigma n \leq -\delta_x(T)d_m^2. \]

If $\lim_{n \to \infty} d_n = 0$, it follows that $\lim_{n \to \infty} \Sigma n = a$ exists. Then if we use $T_v^{(n)}$ as the $T_h'$ in (15) and let $n \to \infty$, we get

(20) \[ 0 \leq \sum_i j(T_i^K) - a \leq -\delta_x(T)d^2 \]

for the arbitrary partition into triangles $T_i$. The definition of area in [3, pp. 70ff.] may—as remarked there—for $T$ be written
\[ j(T) = \inf \lim_{q \to \infty} \sum \hat{j}(T^q) \]

where the \( \lim \) is taken for an arbitrary sequence of partitions \( P^q \) into triangles \( T^q \) with diameters \( d_q \), with \( \lim_{q \to \infty} d_q = 0 \), and the inf is then taken over all such sequences of partitions. Now it follows from (20) both that \( j(T) = a \) and that the inequality of the theorem holds.

**Corollary.** Since, as just observed, \( j(T) = \lim_{n \to \infty} \sum \hat{j}(T^n) \), application of lemma 3 to the partitions \( P_n \) yields, if the diameter of \( T \) is \( d < |K|^{-1} \), the following estimate for \( j(T) - j(T^K) \):

\[ \delta_K(T)d^2 \leq j(T) - j(T^K) \leq 0 . \]

**Remark.** The inequalities given here in the theorem, the corollary and the lemmas are not the best possible. E.g. we have not made use of the factor \( \frac{1}{4} \) obtained in the proof of lemma 1 for \( K \geq 0 \). Also the restriction \( d < |K|^{-1} \) might be weakened.

**Application.** We can now prove that a 2-dimensional \( R_K \) (and thus also an \( R_K \)-space) is of specific curvature \( \leq K \) in the sense of [2, p.431], that is,

\( \omega(E) \leq K j(E) \)

holds for any region \( E \) in the \( R_K \). We begin with the case \( K \geq 0 \). According to the definition of curvature [2, p. 496], we have

\( \omega(E) \leq \omega^+(E) = \sup \sum \delta(T_i)_i \),

where the supremum is taken over all sets of non-overlapping triangles \( T_i \) which are contained in \( E \), and \( \delta(T_i) \) as above is the excess of \( T_i \). For any \( T_i \) we have

\[ \delta(T_i) - K j(T_i) = \delta(T_i) - K j(T_i^K) + K[j(T_i^K) - j(T_i)] \]

\[ = \delta_K(T_i) + K[j(T_i^K) - j(T_i)] \]

\[ \leq \delta_K(T_i) - K \delta_K(T_i)d_i^2 \]

\[ = \delta_K(T_i)(1 - K d_i^2) , \]

because of our corollary above. If the diameter of \( E \) is \( < K^{-1} \), the same holds for \( d_i \) (the diameter of \( T_i \)). Then \( 1 - K d_i^2 > 0 \), and as \( \delta_K(T_i) \leq 0 \) by the definition of an \( R_K \) [3, p. 36], we get \( \delta(T_i) \leq K j(T_i) \) and

\[ \sum \delta(T_i) \leq K \sum j(T_i) \leq K j(E) , \]

which proves (6) for all sufficiently small \( E \). A larger region \( E \) can e.g. by a few geodesics \( g_m \) be divided into sufficiently small parts \( E_n \). Since
the set function $\omega_K = \omega - Kj$ is completely additive and $\omega_K(E_n) \leq 0$, we have

$$\omega_K(E) = \sum_n \omega_K(E_n) + \sum_m \omega_K(g_m) \leq \sum_m \omega_K(g_m) = \sum_m \omega(g_m).$$

This is $\leq 0$, because $\omega(g) \leq 0$ for any geodesic $g$ (cf. [2, p. 498]).

If $K \leq 0$, we have for any triangle $T$

$$\delta(T) = \delta_T(T) + Kj(T_{ik}) \leq \delta_T(T) \leq 0,$$

according to the definition of an $R_K$. The positive part of the curvature $\omega^+$ is thus identically 0 (cf. (21)), and the definition of curvature [2, p. 496] for a region $E$ is simplified to

$$\omega(E) = -\omega^-(E) = \inf \sum_i \delta(T_i),$$

where the infimum is taken over all sets of non-overlapping triangles $T_i$ which are contained in $E$. From the premise $\delta_T(T_i) \leq 0$ we get

$$\delta(T_i) \leq Kj(T_{ik}) \leq Kj(T_i),$$

because $j(T_i) \leq j(T_{ik})$ (theorem 1, p. 71, in [3]). This yields

$$\inf \sum_i \delta(T_i) \leq \inf \sum_i Kj(T_i) = K \cdot \sup \sum_i j(T_i).$$

But, by the usual definition of $j(E)$, we have $\sup \sum_i j(T_i) = j(E)$, and hence

$$\omega(E) = \inf \sum_i \delta(T_i) \leq K \cdot \sup \sum_i j(T_i) = Kj(E),$$

which proves our statement.

REFERENCES

4. A. Beurling, Sur la géométrie métrique des surfaces à courbure totale $\leq 0$, Medd. Lunds Univ. Mat. Sem., Supplementbd. tillägnat Marcel Riesz (1952), 7–11.