MATH. SCAND. 8 (1960), 333–338

LINEAR INEQUALITIES AND POSITIVE EXTENSION OF LINEAR FUNCTIONALS

OTTE HUSTAD

In [4, p. 124] K. Fan has proved a consistency condition for a system of linear inequalities in a normed space. In [5, Proposition 6] we have extended his result to a locally convex Hausdorff space. Later on we have found that both these results in the main are contained in the paper [6] of S. Mazur and W. Orlicz. These authors use linear inequalities to handle the problem of positive extension of linear functionals. In the present paper we have adopted this point of view. To be more specific, our results are as follows. Theorem 1 gives in terms of neighbourhoods a consistency condition for a system of linear inequalities in a locally convex space. In Theorem 2 we give a necessary and sufficient condition that a linear functional admits a positive and continuous extension. This condition stresses the properties of the nullspace of the functional, whereas an earlier known extension condition ([6, 2.72.], [2, Théorème 2], [8, Theorem 4.4]) emphasizes the properties of a neighbourhood of the positive cone. In Proposition 1 we characterize those subspaces F which have the property that every positive (and continuous) linear functional defined on F, admits a positive and continuous extension. If we choose in Proposition 2 the finest locally convex topology, we obtain a characterization of "pyramide stricte" in the terminology of A. Bastiani [1, p. 282]. In the finite dimensional case this proposition, in a slightly different form, has been proved by H. Mirkil [7, p. 3]. The last three propositions are connected with the subject matter in [5], and make use of some results in this paper.

Notation. E shall always denote a real locally convex topological vector space, and P a convex cone in E with vertex 0. Hyperplanes shall always contain 0. Otherwise we use the same notation as in [5].

Theorem 1. *Let the two families* $\{x_\gamma\}_\Gamma \subset E$ *and* $\{\alpha_\gamma\}_\Gamma \subset R$ *be given. Then the linear inequality system*

$$\text{(A)} \qquad f(x_\gamma) \geqq \alpha_\gamma, \qquad \gamma \in \Gamma ,$$

Received August 18, 1960.

is consistent (*i.e. there exists an* $f \in E'$ *which satisfies* (A)), *if and only if there exists a* $V \in \mathscr{V}_E(0)$ *such that*

$$\sum_{i=1}^{n} \lambda_i \alpha_{\gamma_i} < 1$$

whenever

$$\sum_{i=1}^{n} \lambda_i x_{\gamma_i} \in V ,$$

where $\lambda_i \geqq 0$ *and* $\gamma_i \in \Gamma$.

PROOF. If $f \in E'$ satisfies (A), it suffices to choose $V \in \mathscr{V}_E(0)$ such that $|f(x)| < 1$ whenever $x \in V$. Conversely, suppose that $V \in \mathscr{V}_E(0)$ satisfies the condition of the theorem. We may assume without loss of generality that V is closed, convex and symmetric. Let p be the Minkowski functional of V. Thus

$$V = \{x: \ p(x) \leqq 1\} .$$

Hence if

$$p\left(\sum_{i=1}^{n} \lambda_i x_{\gamma_i}\right) \leqq 1$$

with $\lambda_i \geqq 0$ and $\gamma_i \in \Gamma$, then

$$\sum_{i=1}^{n} \lambda_i \alpha_{\gamma_i} < 1 .$$

Since p is a seminorm, it follows that

$$p\left(\sum_{i=1}^{n} \lambda_i x_{\gamma_i}\right) \geqq \sum_{i=1}^{n} \lambda_i \alpha_{\gamma_i}$$

whenever $\lambda_i \geqq 0$ and $\gamma_i \in \Gamma$. Using the theorem of Mazur and Orlicz [6, 2.41] the conclusion follows.

THEOREM 2. *Let* F *be a subspace of* E *and let* f *be a linear functional defined on* F. *Then* f *admits a positive and continuous extension defined on* E *if and only if*

$$f\left(\overline{(f^{-1}(0)+P)} \cap F\right) \geqq 0 .$$

PROOF. The case $f=0$ being trivial we can choose $x_0 \in F$ such that $f(x_0) > 0$. Then f admits a positive and continuous extension defined on E if and only if the system

$$\begin{array}{lll} g(x) & \geqq 0, & x \in f^{-1}(0) \\ g(-x) & \geqq 0, & x \in f^{-1}(0) \\ g(p) & \geqq 0, & p \in P \\ g(x_0) & \geqq 1 & \end{array}$$

is consistent. By Theorem 1, this means that there exists a $V \in \mathscr{V}_E(0)$ such that if

$$x+p+\lambda x_0 \in V, \quad x \in f^{-1}(0), \quad p \in P,\ \lambda \geqq 0,$$

then $\lambda < 1$. It is not hard to verify that this means that

$$-x_0 \notin \overline{P+f^{-1}(0)}\,.$$

Hence we have proved that f admits a positive and continuous extension if and only if

$$\{x \in F\colon\ f(x) < 0\} \cap \overline{P+f^{-1}(0)} = \emptyset\,.$$

Since this condition is equivalent to the condition of the theorem, the proof is finished.

COROLLARY 1. *If f is a positive linear functional defined on F and*

$$\overline{(f^{-1}(0)+P)} \cap F = (f^{-1}(0)+P) \cap F\,,$$

then f admits a positive and continuous extension defined on E.

PROOF. Let

$$y \in \overline{(f^{-1}(0)+P)} \cap F = (f^{-1}(0)+P) \cap F\,.$$

Thus we have $y=x+p$, where $x \in f^{-1}(0)$ and $p \in P$. Hence $p=y-x \in F$, and therefore

$$f(y) = f(x)+f(p) = f(p) \geqq 0\,.$$

PROPOSITION 1. *Let F be a subspace of E. Then the following statements are equivalent*:

(i) *Every positive* (*and continuous*) *linear functional defined on F admits a positive and continuous extension defined on E.*

(ii) *If $H \subset F$ is a* (*closed*) *hyperplane relative to F, then*

$$\overline{(H+P)} \cap F = (H+P) \cap F\,.$$

PROOF. That (ii) implies (i) follows readily from Theorem 2, Corollary 1. To prove the converse, suppose that H is a (closed) hyperplane in F, such that

$$\overline{(H+P)} \cap F \neq (H+P) \cap F\,.$$

Choose

$$x \in \overline{(H+P)} \cap F \sim (H+P) \cap F$$

($\sim$ denotes set-theoretic difference). Since $x \notin H$, every $y \in F$ can be uniquely written

$$y = h+\lambda x$$

where $h \in H$. Define f on F by

$$f(y) = f(h+\lambda x) = -\lambda\,.$$

Then $f^{-1}(0)=H$, and hence f is a (continuous) linear functional on F. Suppose that

$$y = h+\lambda x \in P \cap F .$$

Since $\lambda > 0$ implies that

$$x = \lambda^{-1}y-\lambda^{-1}h \in (P+H) \cap F ,$$

we conclude that $f(y)= -\lambda \geqq 0$. Thus f is a positive (and continuous) linear functional on F. On the other hand, since $f(x)= -1$ and

$$x \in \overline{(f^{-1}(0)+P)} \cap F ,$$

it follows from Theorem 2 that f does not admit a positive and continuous extension defined on E.

Proposition 2. *The following two statements are equivalent:*

(i) *Every positive and continuous linear functional defined on some closed subspace of E admits a positive and continuous extension defined on E.*

(ii) *If M is a closed subspace of E, then $M+P$ is closed.*

Proof. That (ii) implies (i) follows readily from Theorem 2, Corollary 1. To prove the converse, suppose that M is a closed subspace of E and that

$$x \in \overline{M+P} \sim M+P .$$

Let F be the subspace generated by M and x. Then F is closed [3, p. 28]. Define f on F by

$$f(m+\lambda x) = -\lambda .$$

The rest of the argument is similar to that in the proof of Proposition 1 and is omitted.

Proposition 3. *Every positive and continuous linear functional defined on some subspace of E admits a positive and continuous extension defined on E if and only if the following two conditions hold:*

(i) $P \cap \overline{F} \subset \overline{P \cap F}$, *whenever F is a subspace of E;*

(ii) *$M+P$ is closed whenever M is a closed subspace of E.*

Proof. In [5, Proposition 8] we have proved that if G is a dense subspace of E, then every positive and continuous linear functional defined on G admits a positive and continuous extension defined on E if and only if

$$P \subset \overline{P \cap G} .$$

Since any subspace F of E is dense in $\overline{F}$, we can apply this result to the cone $P \cap \overline{F}$. Thus every positive and continuous linear functional defined on F admits a positive and continuous extension defined on $\overline{F}$ if and only if

$$P \cap \overline{F} \subset \overline{P \cap \overline{F} \cap F \cap \overline{F}} = \overline{P \cap F \cap \overline{F}} ,$$

or equivalently

$$P \cap \overline{F} \subset \overline{P \cap F} .$$

The rest of the argument follows from Proposition 2.

PROPOSITION 4. *The following statements are equivalent*:

(i) *Every positive linear functional defined on E is continuous*;

(ii) *$H+P$ is closed whenever H is a hyperplane in E*;

(iii) *$H+P=E$ whenever H is a dense hyperplane in E*;

(iv) *$P \not\subset H$ and $(p+P) \cap H \neq \emptyset$ whenever H is a dense hyperplane in E and $p \in P$*;

(v) *$f(P)=R$ ($=$the set of all real numbers) whenever f is a discontinuous linear functional on E.*

PROOF. By Proposition 1 with $F=E$, (i) implies (ii). Since

$$\overline{H+P} \supset \overline{H} ,$$

it is clear that (ii) implies (iii). Assume $H+P=E$. Then $P \not\subset H$. Further, if $p \in P$, there exists a $h \in H$ and a $q \in P$ such that $-p=h+q$. Hence

$$-h = p+q \in (p+P) \cap H .$$

This shows that (iii) implies (iv). If f is discontinuous then $f^{-1}(0)$ is a dense hyperplane. Assuming (iv) we can therefore find a $p \in P$ such that $f(p) \neq 0$, and a $q \in P$ such that $f(p+q) = f(p)+f(q)=0$. Thus (iv) implies (v). That (v) implies (i) is evident.

PROPOSITION 5. *If $E=P-P$, then (E, P) is an extension couple if and only if P is rich.*

PROOF. According to [5, Proposition 7] we have only to prove that if $E=P-P$ and P is rich, then every positive linear functional on E is continuous. Thus, by the equivalence of (i) and (iv) in Proposition 4, it will suffice to prove that if f is a linear functional on E such that $P \subset f^{-1}(0)$, then f is continuous. Having $E=P-P$, the relation $P \subset f^{-1}(0)$ implies in fact $f=0$.

REFERENCES

1. A. Bastiani, *Cones convexes et pyramides convexes*, Ann. Inst. Fourier Grenoble 9 (1959), 249–292.
2. H. Bauer, *Sur le prolongement des formes linéaires positives*, C. R. Acad. Sci. Paris 244 (1957), 289–292.

3. N. Bourbaki, *Espaces vectoriels topologiques*, Chap. 1–2 (Act. Sci. Ind. 1189), Paris, 1953.
4. K. Fan, *On systems of linear inequalities*, *Linear inequalities and related systems* (Edited by H. W. Kuhn and A. W. Tucker; Ann. of Math. Studies 38), Princeton, 1956, 99–156.
5. O. Hustad, *On positive and continuous extension of positive functionals defined over dense subspaces*, Math. Scand. 7 (1959), 392–404.
6. S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires* II, Studia Mathematica 13 (1953), 137–179.
7. H. Mirkil, *New characterization of polyhedral cones*, Canadian J. Math. 9 (1957), 1–4.
8. I. Namioka, *Partially ordered linear topological spaces* (Mem. Amer. Math. Soc. 24), Providence, 1957.

UNIVERSITY OF OSLO, NORWAY