DISTRIBUTIONS IN Variant UNDER AN ORTHOGonal GROUP OF ARBITRARY SIGNATURE

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1. Introduction.

Let $B = B(u, v)$ be a symmetric real bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$ and let $\mathcal{L}$ be the group of all linear transformations leaving $B$ invariant. A distribution $T(u)$ is said to be invariant under $\mathcal{L}$ if

$$T(\Lambda u) = T(u)$$

for every $\Lambda$ in $\mathcal{L}$. It is very easy to describe e.g. all invariant $T$ with supports at $u = 0$. They are of the form $P(\Box)\delta(u)$ where $\Box$ is Laplace’s operator $B^{-1}(D, D)$ ($D = (\partial/\partial u_1, \ldots, \partial/\partial u_n)$) and $P$ is a polynomial. They span a linear space which we shall call $\mathcal{L}'_0$. Let $\mathcal{L}'$ be the space of all invariant distributions. A rather complete description of $\mathcal{L}'$ has been given by Methée [3] [4], when $B$ has Lorentz signature and by de Rham [6] [7], for general indefinite signature. They show in particular that outside $u = 0$ every $T \in \mathcal{L}'$ has the form

(1) $$\langle T, f \rangle = \langle F, Nf \rangle,$$

where $f$ is any function in $\mathcal{D}(\mathbb{R}^n)$ which vanishes in a neighbourhood of $u = 0$, $F$ is a unique distribution on the real line and $Nf$ is the mean value

$$(Nf)(\tau) = \int \delta(\tau - B(u, u))f(u) \, du,$$

which belongs to $\mathcal{D}(\mathbb{R})$. (We use Schwartz’s notations. $\mathcal{D}$ is the set of infinitely differentiable functions with compact supports.) When $f$ does not vanish at the origin, $Nf$ becomes singular for $\tau = 0$, but has an expansion around $\tau = 0$ in powers of $\tau$ and a suitable additional set of singular functions. The singular expansion coefficients are linear invariant functionals $\langle S, f \rangle$ of $f$ with support at $u = 0$, that is, every $S$ belongs to $\mathcal{L}'_0$ and it turns out that $\mathcal{L}'_0$ is spanned by the distributions $S$.

According to Gårding and Roos (see [2]) a more concise description of $\mathcal{L}'$ can be obtained by putting a suitable linear topology on $H = N\mathcal{D}$. Then to every $T \in \mathcal{L}'$ there is a unique element $F$ in the dual $H'$ of $H$ such

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that (1) holds. More generally the adjoint mapping \( N' \) is a linear homeomorphism of \( H' \) onto \( \mathcal{L}' \). In other words the space \( H' \) gives a parametrization of \( \mathcal{L}' \). Gårding and Roos proved this for the Lorentz group. The main purpose of this paper is to prove the same result when \( B \) has the signature \( p, q \) with \( p + q = n, p \geq 2, q \geq 2 \). We note in passing that it holds also when \( B \) is definite. In this case the space \( H = ND \) is very simple. Changing if necessary \( B \) to \( -B \) we can assume that \( B \) is positive definite. Then \( H \) consists of all functions \( \tau^{-1} f(\tau) \) where \( f \) is infinitely differentiable for \( \tau \geq 0 \). Its dual can be identified with all distributions in \( \tau \) with supports in \( \tau \geq 0 \).

We have assumed that \( \mathcal{L} \) is the entire group leaving \( B \) invariant. Let \( \mathcal{L}_1 \) be the connected component of \( \mathcal{L} \) that contains the unit element and \( \mathcal{L}_1' \supset \mathcal{L}' \) the corresponding space of invariant distributions. It is easy to see that \( \mathcal{L}_1' = \mathcal{L}' \) except when \( B \) has Lorentz signature (see remark p. 13). Although this case does not concern us, we mention that then \( \mathcal{L}_1' \) is the direct sum of \( \mathcal{L}' \) and a space \( \mathcal{L}_- \) of odd invariant distributions with the property that

\[
T(Au) = \varepsilon(A) T(u),
\]

where \( \varepsilon(A) = -1 \) if \( A \) reverses time, \( \varepsilon(A) = 1 \) otherwise. The space \( \mathcal{L}_- \) can be obtained in the same way as \( \mathcal{L}' \) by replacing \( N \) by

\[
(N_- f)(\tau) = \int \delta(\tau - B(u, u)) \text{sgn} B(u, v) f(u) \, du
\]

where \( v \) is any time-like vector. The space \( N_- D \) consists of all functions of \( \tau \) with compact supports which vanish for \( B(v, v) \tau < 0 \) and are infinitely differentiable for \( \tau B(v, v) \geq 0 \) (Gårding and Roos, see [2]).

In outline, our paper runs as follows. In section 2 we introduce the infinitesimal rotations and prove a lemma which we need in section 5. In section 3 we describe and topologize some function spaces, \( H_{s,m} \), where \( s \) assumes four values and \( m \) all integral values \( \geq 1 \). In section 4 we prove that \( N \) is a continuous surjective mapping

\[
\mathcal{D}(R^n) \rightarrow H = H_{s,m},
\]

where \( s \) and \( m \) depend on \( p \) and \( q \). The same is true if we replace \( \mathcal{D} \) by the space \( \mathcal{P} \) of all infinitely differentiable functions which together with their derivatives decrease faster than any negative power of \( |u| \), and modify \( \mathcal{L}' \) and \( H_{s,m} \) accordingly. In section 5 we show that in both cases \( N' \) is a linear homeomorphism of \( H'_{s,m} \) onto \( \mathcal{L}' \). The rest of the paper is devoted to some applications of this result. Let \( G \) be a linear operator from \( \mathcal{L}' \) to \( \mathcal{L}' \). Transported to \( H' \) it becomes \( \Gamma = N' GN'^{-1} \).
When \( G = □ \) is Laplace’s operator, \( \Gamma \) is the adjoint of the differential operator
\[
D = 4(\tau D_\tau^2 + \frac{1}{6}(n - 4)D_\tau), \quad D_\tau = d/d\tau,
\]
which maps \( H \) into itself and has the property that \( N □ f = D N f \) \((f \in \mathcal{D}(\mathbb{R}^n))\). Using \( D \) it is easy to write down all invariant fundamental solutions of \( □ \) (de Rham [7]). We do this in section 6 and in section 8 we prove that the equation \( P(□) T = S \) where \( P \) is an arbitrary polynomial has a solution \( T \) in \( \mathcal{L}' \) for every \( S \) in \( \mathcal{L}' \). When \( B \) has Lorentz signature this was shown by Methée [3]. Finally in section 7 we get an explicit expression for \( \Gamma \) when \( G \) is the Fourier transform \( \mathcal{F} \). In the Lorentz case, Fourier-transforms of invariant distributions were studied by Methée [4].

Since the use of the homeomorphism \( N' \) makes the subject very clear and simple, I have chosen to make the paper self-contained although this leads to considerable overlappings with the papers by de Rham and Methée.

The subject of this paper was suggested to me by professor Lars Gårding. I wish to express my gratitude to him for his interest and valuable advice.

2. Infinitesimal rotations.

We choose a coordinate system so that
\[
B(u, v) = \sum_{i=1}^{p} x_i x'_i - \sum_{k=1}^{q} y_k y'_k = xx' - yy',
\]
where \( u = (x, y) \) and \( v = (x', y') \), and we put
\[
B(u, v) = uv.
\]
Every \( \Lambda \in \mathcal{L} \) can be written as
\[
\Lambda = \Lambda^x \Lambda^y \Lambda_\theta,
\]
where \( \Lambda^x \) and \( \Lambda^y \) belong to \( \mathcal{L} \) and leave \( y \) resp. \( x \) fixed and \( \Lambda_\theta \) is defined by
\[
x'_j = x_j \quad \text{when} \quad j \neq i, \quad y'_l = y_l \quad \text{when} \quad l \neq k,
\]
\[
x'_i = x_i \cosh \theta + y_k \sinh \theta,
\]
\[
y'_k = x_i \sinh \theta + y_k \cosh \theta.
\]
The group \( \mathcal{L} \) consists of four connected components \( \mathcal{L}_{++} (= \mathcal{L}_1), \mathcal{L}_{+-}, \mathcal{L}_{-+}, \mathcal{L}_{--} \), where the transformations of \( \mathcal{L}_{+-} \), are characterized by \( \det \Lambda^x = 1 \) and \( \det \Lambda^y = -1 \).

Let
\[ L_{ij}^x = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \]
\[ L_{kl}^y = y_k \frac{\partial}{\partial y_l} - y_l \frac{\partial}{\partial y_k}, \]
\[ L_{ik} = x_i \frac{\partial}{\partial y_k} + y_k \frac{\partial}{\partial x_i} \]
be the infinitesimal rotations. We now have the following lemma:

**Lemma 2.1.** \( T \) belongs to \( \mathcal{L}^\prime_{++} \) if and only if

(2) \[ L_{ij}^x T = L_{kl}^y T = L_{ik} = 0 \quad \text{for} \quad 1 \leq i, j \leq p, 1 \leq k, l \leq q, \]

**Proof.** Define transformations \( \Lambda^x_0 \) by

\[
\begin{align*}
x_k' &= x_k \quad \text{if} \quad k + i, k + j, \quad y_l' = y_l \quad \text{for every} \quad l, \\
x_i' &= x_i \cos \theta + x_j \sin \theta, \\
x_j' &= -x_i \sin \theta + x_j \cos \theta.
\end{align*}
\]

\( T \) is invariant under all these transformations \( \Lambda^x_0 \), that is,

\[ \langle T, f(u) \rangle = \langle T, f(\Lambda^x_0 u) \rangle \quad \text{for every} \quad f \in \mathcal{D}(\mathbb{R}^n) \]

if and only if

\[
\frac{d}{d\theta} \langle T, f(\Lambda^x_0 u) \rangle = \left\langle T, \frac{\partial}{\partial \theta} f(\Lambda^x_0 u) \right\rangle = 0
\]

for every \( f \in \mathcal{D}(\mathbb{R}^n) \). But as

\[
\frac{\partial}{\partial \theta} f(\Lambda^x_0 u) = (L_{ij}^x f)(\Lambda^x_0 u)
\]

this is equivalent to \( L_{ij}^x T = 0 \).

In the same way we define transformations \( \Lambda^y_0 \) and prove that \( T \) is invariant under these transformations if and only if \( L_{kl}^y T = 0 \) for every \( k \) and \( l \).

Similarly we can prove that \( T \) is invariant under all the transformations \( \Lambda_0 \) defined above if and only if \( L_{ik} = 0 \) for every \( i \) and \( k \). Now if \( T \in \mathcal{L}^\prime_{++} \), \( T \) is invariant under all the transformations \( \Lambda^x_0, \Lambda^y_0 \) and \( \Lambda_0 \) and consequently (2) holds. As an arbitrary transformation \( \Lambda \) in \( \mathcal{L}^\prime_{++} \) can be written as a product of \( \Lambda^x_0, \Lambda^y_0 \) and \( \Lambda_0 \) we see that (2) implies that \( T \in \mathcal{L}^\prime_{++} \).

**Remark.** Since \( \mathcal{L}' \subset \mathcal{L}^\prime_{++} \) it is clear \( T \in \mathcal{L}' \) implies (2). Later we shall see that in fact \( \mathcal{L}' = \mathcal{L}^\prime_{++} \).
3. Some function spaces.

We are going to define spaces of functions \( \varphi, \psi, \ldots \) of one variable \( \tau \) which are regular for \( \tau \neq 0 \) and have a singularity for \( \tau = 0 \) defined in terms of one of four functions \( \gamma = \gamma_s(\tau) \) labelled by an index \( s \) and given by

\[
\theta(\tau), \quad \theta(\pm \tau)(\pm \tau)^l, \quad \log|\tau|^{-1}
\]

according as \( s = 1, \pm \frac{1}{2}, \text{ and } l \) respectively. Here \( \theta \) is Heaviside’s function, \( \theta(\tau) = 1 \) if \( \tau \geq 0 \) and \( \theta(\tau) = 0 \) if \( \tau < 0 \). Let \( m \) be a fixed integer \( \geq 1 \) and denote by \( P_v(\tau) \) polynomials of degree \( \leq v \) divisible by \( \tau^m \). In particular \( P_v(0) = 0 \) unless \( v \geq m \). It is obvious that

\[
(3) \quad \gamma(\tau)P_v(\tau) \in C^v \quad \text{if and only if} \quad P_v = 0.
\]

We shall consider functions of class \( C^v \) outside the origin with the property that

\[
(4) \quad \varphi - \gamma P_v \in C^v
\]

at the origin for at least one polynomial \( P_v \). It follows from (3) that \( P_v \) is uniquely determined by \( \varphi \). Further, if \( v < m \), \( P_v = 0 \) so that \( \varphi \) is itself of class \( C^m \) at the origin. It is clear that the coefficients \( A_j(\varphi) \) of \( P_v \),

\[
P_v(\tau) = \sum_m^n A_j(\varphi) \tau^j,
\]

are linear functions of \( \varphi \) and that \( A_j(\varphi) = 0 \) if \( \varphi \in C^j \) at the origin. In particular \( A_j(\varphi) = 0 \) if \( \varphi \) vanishes in a neighbourhood of the origin. We shall find it convenient to write \( P_v \) as

\[
P_v(\tau) = \sum_0^n A_j(\varphi) \tau^j,
\]

where, by definition, \( A_j(\varphi) = 0 \) when \( j < m \). Expanding (4) in a Taylor series around \( \tau = 0 \),

\[
\varphi(\tau) - \gamma(\tau)P_v(\tau) = \sum_0^n B_j(\varphi) \tau^j + o(\tau^n)
\]

we obtain another set \( B_j \) of linear functionals of \( \varphi \) with supports at the origin. Thus every \( \varphi \) with the property (4) has a unique expansion of the form

\[
\varphi(\tau) = \sum_0^n B_j(\varphi) \tau^j + \gamma(\tau) \sum_0^n A_j(\varphi) \tau^j + o(\tau^n).
\]

Now let \( a > 0 \) and let \( C^v_a \) and \( H^v_a \) be the space of all \( \varphi \in C^v \) with support in \( |\tau| \leq a \) and the space of all \( \varphi \) with the property (4) and support in \( |\tau| \leq a \) respectively. It is clear that both spaces decrease when \( v \) increases.
and that $H^v_a \supset C^v_a$. More precisely any $\varphi \in H^v_a$ has a unique decomposition

(5) \[ \varphi = (\varphi - \gamma P_v) + \gamma P_v, \]

where the first term belongs to $C^v_a$. Hence $H^v_a$ is the direct sum of $C^v_a$ and a space of dimension $\max(v-m,0)$. With the norms

(6) \[ |\varphi|_v = \max_{k \leq v} \max_\tau |\varphi^{(k)}(\tau)| \]

and

(7) \[ g_v(\varphi) = |\varphi - \gamma P_v|_v + \sum_0^v |A_j(\varphi)|, \]

$C^v_a$ and $H^v_a$ become Banach spaces. It is clear that the $A_j$ and $B_j$ with $j \leq v$ are continuous on $H^v_a$. Equipped with the norms (6),

\[ C_a = C^\infty_a = \bigcap_{v \geq 0} C^v_a \]

becomes a reflexive Fréchet space. The reflexivity follows from the classical fact that the injections $C^{v+1}_a \rightarrow C^v_a$ are completely continuous. Since, obviously, the injections $H^{v+1}_a \rightarrow H^v_a$ are also completely continuous, the space

\[ H_a = H^\infty_a = \bigcap_{v \geq 0} H^v_a \]

equipped with the norms (7), is also a reflexive Fréchet space. All the $A_j$ and $B_j$ are continuous on $H^v_a$. It is clear that $H_a$ consists of all $\varphi$ with support in $|\tau| \leq a$ which are in $C^\infty$ for $\tau \neq 0$ and for which there exists a formal power series

\[ P(\tau) = \sum_0^\infty a_j \tau^j, \quad a_j = 0 \quad \text{if} \quad j < m, \]

with partial sums $P_v(\tau) = \sum_0^v a_j \tau^j$ such that

(8) \[ \varphi - \gamma P_v \in C^v \quad \text{at} \quad \tau = 0 \quad \text{for all} \quad v. \]

In particular, $a_j = A_j(\varphi)$ so that $P$ is uniquely determined by $\varphi$. The direct decomposition (5) fails to hold in the infinite case. Instead we have

**Lemma 3.1.** $\varphi$ belongs to $H_a$ if and only if

(9) \[ \varphi(\tau) = \varphi_1(\tau) + \gamma(\tau)\varphi_2(\tau)\tau^m, \quad \text{where} \quad \varphi_1, \varphi_2 \in C_a. \]

**Proof.** It is clear that any $\varphi$ of the form (9) is in $H_a$. Conversely, let $\varphi \in H_a$. Then, by a classical result due to Borel, we can find a $\varphi_2 \in C_a$ such that $\tau^m \varphi_2$ has the Taylor series $P$ at the origin. Hence $\varphi - \gamma \tau^m \varphi_2 \in C_a$. 

Finally, we shall define spaces of functions with arbitrarily large supports having the property (4) for all \( v \). This can of course be done in various ways. We choose to define the analogue of L. Schwartz’s space \( \mathcal{D}(R) \) which is the inductive limit (Bourbaki [1, Chap. II p. 61–65]) of the spaces \( C_a \) when, for example, \( a = 1, 2, \ldots \). We let \( H \) be the inductive limit of the spaces \( H_a \) for \( a = 1, 2, \ldots \) . It consists of all \( \varphi \) with compact supports, belonging to \( C^\infty \) for \( \tau \not= 0 \) and having the property (4). We have

**Lemma 3.2.** \( H \) is reflexive.

**Proof.** \( H \) is in fact the strict inductive limit of the \( H_a \) and hence \( H \) is reflexive (Bourbaki [1, Chap. IV, p. 95, Exc. 17 b]).

A complete set of seminorms for \( \mathcal{D}(R) \) can be obtained by putting

\[
h(\varphi) = \sum_k \max_{\tau} |h_k(\tau)\varphi^{(k)}(\tau)|,
\]

where the \( h_k \) are continuous functions with the property that for every compact \( K \) there is a \( \lambda(K) \), such that \( h_k = 0 \) in \( K \) when \( k > \lambda \). Correspondingly, we obtain a complete set of seminorms on \( H \) by choosing a function \( \chi \in \mathcal{D}(R) \) which is 1 in a neighbourhood of \( \tau = 0 \) and putting

\[
g(\varphi) = h(\varphi - \chi \gamma P_v) + \sum_1^\mu |A_j(\varphi)|,
\]

where \( v > \lambda(K) \), \( K = \text{supp} \chi \) and \( \mu \) is an arbitrary integer. Changing \( \chi \), we get an equivalent set of seminorms.

Any element \( F \) in the dual \( H' \) of \( H \) can be described in terms of a distribution and the functionals \( A_j \). We have

**Lemma 3.3.** \( F \) belongs to \( H' \) if and only if \( F \) has the form

\[
\langle F, \varphi \rangle = \langle F_0, \varphi - \chi \gamma P_v \rangle + \sum_0^v c_j A_j(\varphi),
\]

where \( F_0 \) belongs to \( \mathcal{D}'(R) \) and the order of \( F_0 \) on the support of \( \chi \) is less than \( v \).

**Proof.** Clearly \( F \in H' \) if \( F \) is defined by (10). Let \( F \in H' \) and let \( F_0 \) be the restriction of \( F \) to \( \mathcal{D}(R) \). Then \( F_0 \in \mathcal{D}'(R) \), and if \( v \geq \) the order of \( F_0 \) on the support of \( \chi \), we define \( F_1 \) by

\[
\langle F_1, \varphi \rangle = \langle F_0, \varphi - \chi \gamma P_v \rangle.
\]

Then we have \( F_1 = F \) when \( A_j(\varphi) = 0, j \leq v \), which implies

\[
F = F_1 + \sum_0^v c_j A_j.
\]
Corollary. Any \( F \in H' \) with support at \( \tau = 0 \) has the form

\[
\sum c'_j B_j + \sum c''_j A_j,
\]

where the sums are finite.

Remark. We can also define \( H \) as the analogue of the space \( \mathcal{S} \) (Schwartz [8]), i.e. the space of all functions \( \varphi \in C^\infty \) for which the norms

\[
\sum_{j, k \leq \nu} |\tau^j \varphi^{(k)}(\tau)|_0
\]

are finite. The lemmas 3.1–3.3 are still true.

Remark. When we want to exhibit that \( H \) and \( H' \) depend on \( s \) and \( m \) we write \( H_{s, m} \) and \( H'_{s, m} \).

4. The mapping of \( \mathcal{D}(R^n) \) onto \( H_{s, m} \).

Let \( g \in \mathcal{D}(R) \) and \( f \in \mathcal{D}(R^n) \) and consider the integral

\[
\int g(uu) f(u) \, du.
\]

Make a change of variables such that

\[
2\sigma = uu, \quad 2\rho = xx + yy, \quad x = (\rho + \sigma)^i w_x, \quad y = (\rho - \sigma)^i w_y,
\]

where \( w_x \) belongs to the \((p-1)\)-dimensional sphere, \( S_{p-1} \), and \( w_y \in S_{q-1} \). Then the integral becomes

\[
\int_{e^2 |\sigma|} g(\sigma) Mf(\rho, \sigma)(\rho + \sigma)^{\bar{\rho}}(\rho - \sigma)^{\bar{\sigma}} \, d\rho \, d\sigma,
\]

where

\[
(Mf)(\rho, \sigma) = \int f((\rho + \sigma)^i w_x, (\rho - \sigma)^i w_y) \, dw_x \, dw_y.
\]

dw_x and dw_y are the surface elements of \( S_{p-1} \) and \( S_{q-1} \) and \( \bar{\rho} = \frac{1}{2}(p-2) \) and \( \bar{\sigma} = \frac{1}{2}(q-2) \).

Hence letting \( g \) approach \( \delta(\tau - \sigma) \) we get

\[
\int \delta(uu - \tau) f(u) \, du = \int_{e^2 |\sigma|} Mf(\rho, \tau)(\rho + \tau)^{\bar{\rho}}(\rho - \tau)^{\bar{\sigma}} \, d\rho \, d\sigma.
\]

We observe that \( \delta(uu - \tau) \in \mathcal{S}' \) and that the support of \( \delta(uu - \tau) \) is the hyperboloid \( uu = \tau \).

Define the mapping \( N \) by

\[
(Nf)(\tau) = \int \delta(uu - \tau) f(u) \, du \quad \text{for every} \quad f \in \mathcal{D}(R^n).
\]

Put

\[
\Omega = R^n - \{0\}, \quad \Omega = \{ (\rho, \sigma); \ e^2 |\sigma| \} \quad \text{and} \quad Q_0 = \Omega - \{0\}.
\]
Lemma 4.1. $M$ defines linear surjective continuous mappings

\begin{align}
(11) & \quad \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(Q), \\
(12) & \quad \mathcal{D}(\Omega) \to \mathcal{D}(Q_0)
\end{align}

Proof. (12) follows from (11). We prove (11).

The only thing which is difficult to prove is that $Mf$ has continuous derivatives of any order in $(0,0)$ which evidently is equivalent to proving the same property for $M_1f$, where

$$M_1f(\xi, \eta) = \int f(\xi^i w_x, \eta^j w_y) \, dw_x \, dw_y = \psi(\mu, v)$$

with $\mu^2 = \xi$, $v^2 = \eta$. We observe that $\psi \in C^\infty(\mathbb{R}^2)$ and that $\psi$ is even in $\mu$ and $v$. We have

$$\frac{\partial}{\partial \xi}(M_1f) = (2\mu)^{-1}(\partial/\partial \mu)\psi = \psi_1(\mu, v),$$

where $\psi_1$ is continuous and even in $\mu$ for $(\partial/\partial \mu)\psi$ is odd in $\mu$. Now it follows by induction that an arbitrary derivative of $M_1f$ is continuous. Furthermore, we have

$$\left(\frac{\partial}{\partial \xi}(M_1f)(0,0) = \frac{1}{2}(\partial/\partial \mu)\psi(0,0)\right).$$

Remark. From the above proof it follows that

$$D_\xi^\alpha D_\eta^\beta (Mf)(0,0) = \sum_{|\alpha| \leq 2(\beta + \gamma)} c_\alpha(D_\alpha f)(0),$$

where some $c_\alpha \neq 0$ with $|\alpha| = 2(\beta + \gamma)$.

Lemma 4.2. $N$ is a linear continuous surjective mapping $\mathcal{D}(\Omega) \to \mathcal{D}(R)$.

Proof. According to Lemma 4.1 it is sufficient to prove that the mapping $L: \mathcal{D}(Q_0) \to \mathcal{D}(R)$ defined by

$$Lg(\tau) = \int g(q, \tau)(q + \tau)\bar{p}(q - \tau)\bar{q} \, dq$$

is a linear continuous surjective mapping. As $g(q, \sigma) = 0$ in a neighbourhood of $(0,0)$, it is easily seen that $Lg \in \mathcal{D}(R)$. Clearly $L$ is linear and continuous.

Let $\varphi \in \mathcal{D}(R)$ and put $K = \text{supp} \varphi$. Let $I \subset R$ be a compact interval so that $I \times K$ is contained in the interior of $Q_0$ and let $\psi \in \mathcal{D}(I)$ so that

$$\int \psi(q) \, dq = 1.$$

If

$$h(q, \sigma) = \psi(\sigma)\varphi(q + \sigma)^{-\bar{p}}(q - \sigma)^{-\bar{q}}$$

we have $Lh(\tau) = \varphi(\tau)$ and hence $N$ is surjective.

Now we can prove the following basic lemma.
Lemma 4.3. \(N\) is a linear, continuous, surjective mapping \(\mathcal{D}(R^n) \rightarrow H_{s,m}\) where \(m = \left[\frac{1}{4}(n-2)\right]\) and

I) \(s = 1\) if \(p\) and \(q\) both are even,

II) \(s = +\frac{1}{2}\) if \(p\) is odd and \(q\) is even,

II') \(s = -\frac{1}{2}\) if \(p\) is even and \(q\) is odd,

III) \(s = l\) if \(p\) and \(q\) both are odd.

Remark. We observe that II' follows from II by interchanging \(x\) and \(y\).

Proof. At first we show that \(N(\mathcal{D}(R^n)) \subset H_{s,m}\). Clearly \(Nf\) has compact support and is of class \(C^\infty\) for \(\tau \neq 0\). Put \((Mf)(\sigma, \omega) = \mathcal{h}(\omega + \sigma, \omega - \sigma)\) with \(h \in \mathcal{D}(R^+ \times R^+)\). We examine \(Nf\) in \(\mathcal{C} = \{\tau; \ |\tau| \leq \frac{1}{2}\}\). It is easily seen that

\[\int_{\frac{1}{2}}^\infty h(q + \tau, q - \tau)(q + \tau)^{\bar{p}}(q - \tau)^{\bar{q}}d\tau \in C^\infty(\mathcal{C}).\]

I. \(p\) and \(q\) both even. It is easily seen that

\[\int_{\frac{1}{2}}^1 h(q + \tau, q - \tau)(q + \tau)^{\bar{p}}(q - \tau)^{\bar{q}}d\tau \in C^{\bar{p}+\bar{q}}(\mathcal{C}).\]

If we put \(h_{\beta,\gamma} = D_1^\beta D_2^\gamma h\) we get by developing in a Taylor series and integrating by parts

\[Nf(\tau) = \sum_{\beta + \gamma \leq v} \binom{\beta + \gamma}{\beta} h_{\beta,\gamma}(0,0) \int_{\frac{1}{2}}^1 (q + \tau)^{\bar{p}+\beta}(q - \tau)^{\bar{q}+\gamma}d\tau + w\]

\[= \theta(\tau) \sum_{\beta + \gamma \leq v} a_{\beta,\gamma} \tau^{\bar{p}+\bar{q}+\beta+\gamma+1} h_{\beta,\gamma}(0,0) + \sum_{\beta + \gamma \leq v} h_{\beta,\gamma}(0,0) w_{\beta,\gamma} + w,\]

where \(w \in C^{v+1}(\mathcal{C})\) and \(w_{\beta,\gamma} \in C^\infty(\mathcal{C})\) is independent of \(f\), and

\[a_{\beta,\gamma} = \frac{\Gamma(\bar{p} + \beta + 1) \Gamma(\bar{q} + \gamma + 1)}{\Gamma(\bar{p} + \bar{q} + \beta + \gamma + 2)} \binom{\beta + \gamma}{\beta} \bar{p}^{\bar{p}+\bar{q}+\beta+\gamma+1}.\]

Hence \(N(\mathcal{D}(R^n)) \subset H_{1,m}\) and the expression for \(Nf\) shows that

(13) \[A_{m+k}(Nf) = \sum_{|\alpha| \leq 2k} c_\alpha(D_\alpha f)(0),\]

where \(c_\alpha \neq 0\) for some \(\alpha, |\alpha| = 2k\).
II. $p$ odd, $q$ even. It is easily seen that

$$\int_{|\tau|}^{1} h(q+\tau, q-\tau)(q+\tau)^{i+1}(q-\tau)^{j} \, dq \in \mathcal{C}^{i+j+1}(\mathcal{O}).$$

By developing in a Taylor series and integrations by parts we get

$$Nf(\tau) = \sum_{\beta+\gamma \leq \nu} \left( \frac{\beta+\gamma}{\beta} \right) h_{\beta, \gamma}(0, 0) \int_{|\tau|}^{1} (q+\tau)^{\beta+\gamma}(q-\tau)^{\beta+\gamma} \, dq + w$$

$$= \theta(\tau)(2\pi)^{\frac{1}{2}} \sum_{\beta+\gamma \leq \nu} a_{\beta, \gamma} \tau^{\beta+\gamma} h_{\beta, \gamma}(0, 0) + \sum_{\beta+\gamma \leq \nu} h_{\beta, \gamma}(0, 0) w_{\beta, \gamma} + w,$$

where $w \in \mathcal{C}^{v+1}(\mathcal{O})$ and $w_{\beta, \gamma} \in \mathcal{C}^{\infty}(\mathcal{O})$ is independent of $f$. Hence $N(\mathcal{D}(\mathbb{R}^n)) \subset H_{+m}$ and the expression for $Nf$ gives again (13).

III. $p$ and $q$ both odd. It is easily seen

$$\int_{|\tau|}^{1} h(q+\tau, q-\tau)(q+\tau)^{i+1}(q-\tau)^{j+1} \, dq \in \mathcal{C}^{i+j+1}(\mathcal{O}).$$

We get by developing in a Taylor series and integrating by parts

$$Nf(\tau) = \sum_{\beta+\gamma \leq \nu} \left( \frac{\beta+\gamma}{\beta} \right) h_{\beta, \gamma}(0, 0) \int_{|\tau|}^{1} (q+\tau)^{\beta+\gamma}(q-\tau)^{\beta+\gamma} \, dq + w$$

$$= 2\pi^{-1} \log |\tau|^{-1} \sum_{\beta+\gamma \leq \nu} a_{\beta, \gamma} \tau^{\beta+\gamma} h_{\beta, \gamma}(0, 0) +$$

$$+ \sum_{\beta+\gamma \leq \nu} h_{\beta, \gamma}(0, 0) w_{\beta, \gamma} + w,$$

where $w \in \mathcal{C}^{v+1}(\mathcal{O})$ and $w_{\beta, \gamma} \in \mathcal{C}^{\infty}(\mathcal{O})$ is independent of $f$. Hence $N(\mathcal{D}(\mathbb{R}^n)) \subset H_{+m}$ and the expression for $Nf$ gives again (13).

We shall now show that $N$ is surjective. Let $\varphi \in H_{s,m}$. From the calculations above it follows that there is an $h \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^+)$ so that

$$\int_{|\tau|}^{1} h(q+\tau, q-\tau)(q+\tau)^{\beta}(q-\tau)^{\beta} \, dq$$

has the same singular part as $\varphi$. Take an $f_1 \in \mathcal{D}(\mathbb{R}^n)$ so that $Mf_1 = h(q+\tau, q-\tau)$. We have $Nf_1 - \varphi \in \mathcal{D}(\mathbb{R})$. By Lemma 4.2 there is an $f_2$ in $\mathcal{D}(\mathcal{O})$ so that $Nf_2 = Nf_1 - \varphi$. We now have $Nf = \varphi$ with $f = f_1 - f_2 \in \mathcal{D}(\mathbb{R}^n)$.

Proof of the continuity of $N$: Let $f \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^n)$. It is easily seen that supp$(Nf_j)$ is contained in a fixed compact set $K \subset \mathbb{R}$. From (13) it follows that $A_{k}(Nf_j) \rightarrow 0$. Furthermore we have with $0 < \delta < 1$
\[ |D_r^\tau(Nf_j) - \gamma_\theta(\tau) \chi(\tau) P_\nu(\tau)| \]
\[ \leq \left| D_r^\tau \left( \sum_{|\beta| + \gamma \leq \nu} c_{\beta, \gamma} h_{j, \beta, \gamma}(\vartheta(q + \tau), \vartheta(q - \tau)) \cdot (q + \tau)^{\overline{\vartheta}(q - \tau)} \overline{\alpha} \overline{\beta} \overline{\alpha} \overline{\beta} d\vartheta + \sum_{\beta + \gamma \leq \nu} h_{j, \beta, \gamma}(0, 0) w_{\beta, \gamma} \right) \right| + \]
\[ + C \sum_{k=0}^\nu |A_k(Nf_j)| , \]
which tends to zero uniformly in $K$ and hence $N$ is continuous.

5. Parametrization of $L'$.

**Lemma 5.1.** If $T \in L'$ there is one and only one distribution $F \in D'(R)$ such that
\[ \langle T, f \rangle = \langle F, Nf \rangle \quad \text{for every } f \in D(\Omega) . \]

**Proof.** After having in a convenient manner introduced new coordinates $\eta(x, y) = (q, \sigma, \theta_x, \theta_y)$ where $2q = xx + yy$ and $2\sigma = uu$, using Lemma 2.1 we can prove that $T \circ \eta$ is independent of $(q, \theta_x, \theta_y)$. Hence we easily get the existence of $F$. (We have here defined $T \circ \eta$ by $\langle T \circ \eta, f \rangle = \langle T, f \circ \eta^{-1} | J(\eta^{-1}) \rangle$, where $J$ denotes the Jacobi determinant; we assume that $\inf |J(\eta)| > 0$ in the set). By use of Lemma 4.2, the uniqueness of $F$ follows from the fact that $N(D(\Omega)) = D(R)$. (Cf. de Rham [6] [7] and for the case $p = 1$ Methée [3].)

**Lemma 5.2.** $T \in L'$ and supp $T \subset \{0\}$ if and only if $T = P(\square) \delta$, where $P$ is a polynomial.

**Proof.** In fact, every such $T$ has the form $Q(D) \delta(u)$ where $Q$ is an invariant polynomial and hence of the form $P(\square) \delta$.

We observe that if $F \in H'$ then $F$ defines a distribution $T \in L'$ by
\[ \langle T, f \rangle = \langle F, Nf \rangle \quad \text{for every } f \in D(R^n) . \]
Put $T = N'F$ where $N'$ is the adjoint mapping to $N$. 

**Lemma 5.3.** $T \in L'$ and supp $T \subset \{0\}$ if and only if $T = \sum c_j N'A_j$, where the sum is finite.

**Proof.** Let $G_z$ be all distributions in question whose orders are $\leq 2v$. Lemma 5.2 shows that $\dim G_z = v + 1$. From (13) it follows that $N'A_{m+k} \in G_z$ for every $k \leq v$ and that the $N'A_{m+k}$ are linearly independent, and hence we have the lemma.

Now we can easily get the parametrization of $L'$. 
THEOREM 5.1. $N'$ is a linear homeomorphism $H'_{s,m} \to \mathcal{L}'$, where $s$ and $m$ depend on $p$ and $q$ as in Lemma 4.3.

PROOF. At first we prove that $N' H'_{s,m} = \mathcal{L}'$. Clearly $N' H'_{s,m} \subset \mathcal{L}'$. Let $T \in \mathcal{L}'$. From Lemma 5.1 it follows that there is a unique $F_0 \in \mathcal{D}'(R)$ such that

$$\langle T, f \rangle = \langle F_0, Nf \rangle \quad \text{for every} \quad f \in \mathcal{D}(\Omega).$$

If $v \geq \text{the order of } F_0 \text{ in } \text{supp } \chi$, we define an extension $F_1$ of $F_0$ by

$$\langle F_1, \varphi \rangle = \langle F_0, \varphi - \gamma_{s}(\tau) \chi(\tau) \sum_{j=0}^{v} A_j(\varphi) \tau^j \rangle \quad \text{for every} \quad \varphi \in H_{s,m}.$$ 

But as $\text{supp} (T - N' F_1) \subset \{0\}$ we have $T - N' F_1 = \sum c_j A_j$ (Lemma 5.3) and consequently $T = N' F$ with $F = F_1 + \sum c_j A_j$.

Now the theorem follows from general theorems if we observe that $\mathcal{L}'$ (see Bourbaki [1, Chap. IV, p. 80, Ex. 9a] and Schwartz [8, I, p. 72]), that the spaces $H'_{s,m}$ (inductive limits of Fréchet spaces) are barrels, and that $N$ is continuous and surjective $\mathcal{D}(R^n) \to H_{s,m}$. For the general theorems see Bourbaki [1, Chap. IV, p. 70, 102–104].

REMARK. Lemma 5.1 holds even if we only suppose $T \in \mathcal{L}'_{++}$ which depends on the fact that $\mathcal{L}'_{++}$ as well as $\mathcal{L}'$ acts transitively on $\Omega$ (which is not true when $p = 1$ or $q = 1$), which implies that the distribution $F$ is independent of the set in which we introduced new variables. As also Lemmas 5.2 and 5.3 hold if we change $\mathcal{L}'$ to $\mathcal{L}'_{++}$, Theorem 5.1 holds if we change $\mathcal{L}'$ to $\mathcal{L}'_{++}$ and consequently $\mathcal{L}' = \mathcal{L}'_{++}$.

REMARK. We put $n = \frac{1}{2}(n-2)$ and $m = [n]$. From the proof of Lemma 4.3 we get

(14') \hspace{1cm} N' A_m = (2 \pi)^{\frac{n+1}{2}} / \Gamma(n+1) \quad \text{when } pq \text{ is even,}

and

(14'') \hspace{1cm} N' A_m = (2 \pi)^{n} / \Gamma(n+1) \quad \text{when } pq \text{ is odd.}

6. Fundamental solutions of $\Box$.

By direct calculation we get

LEMMA 6.1. $N \Box f = DNf$ for every $f \in \mathcal{D}(R^n)$, where

$$D = 4(\tau D \tau^2 + \frac{1}{2}(n-4) D).$$

We make the following

DEFINITION. Define $Pf. \tau^{-v}, v \leq m + 1$, by

$$\langle Pf. \tau^{-v}, f \rangle = \text{p.v.} \int_{-\infty}^{+\infty} \tau^{-v} (f(\tau) - \sum_{0}^{[v]-2} (f^{(j)}(0) \tau^{j-1}) \int_{-\infty}^{+\infty} d\tau$$

for every $f \in H_{s,m}$, where p.v. denotes the principal value of Cauchy.
Now we get the following theorems from the parametrization of $L'$.

**Theorem 6.1** If we put

$$E = \frac{1}{8}(2\pi)^{-\frac{3n}{2}} \Gamma(n + 1)(n - 2)^{-\frac{1}{2}} \text{Pf. } \tau^{-n}$$

when at least one of $p$ and $q$ is even, and

$$E = \frac{1}{16}(2\pi)^{-\frac{3n}{2}} \Gamma(n + 1) B_{\frac{n}{2}}$$

when both $p$ and $q$ are odd, we have $\Box N' E = \delta$.

**Proof.** It is easily seen that if $F \in H_{s, m}'$ and $D'F = A_{m}$ then $\Box(N'F) = k \delta$, where $m = [\frac{n}{2}]$ and $k = k(p, q)$ is given by (14) and $D'$ is the formal adjoint of $D$. When at least one of $p$ and $q$ is even it is immediately verified that $D'E = A_{m}/k$. When both $p$ and $q$ are odd the equality $D'E = A_{m}/k$ follows from the asymptotic development of $D\varphi$ at the origin

$$\frac{1}{4}D\varphi \sim \sum (B_{j}j(j - \frac{n}{2}) + A_{j}(2j - \frac{n}{2})) \tau^{j} + \log |\tau|^{-1} \sum A_{j}(j - \frac{n}{2}) \tau^{j}$$

if

$$\varphi \sim \sum B_{j} \tau^{j} + \log |\tau|^{-1} \sum A_{j} \tau^{j}.$$

**Theorem 6.2.** Put $G = B_{\frac{n}{2}}$ when at least one of $p$ and $q$ is even, and $G = \text{Pf. } \tau^{\frac{n}{2}}$ when both $p$ and $q$ are odd. Every solution $T \in L'$ of $\Box T = 0$ can be written in the form $a N' G + b$.

**Proof.** It is easily seen that $D'F = 0$ if and only if $\Box N' F = 0$. When at least one of $p$ and $q$ is even, the equality $D'G = 0$ follows from the asymptotic development of $D\varphi$ at the origin where $\varphi \sim \sum B_{j} \tau^{j} + \gamma_{s} \sum A_{j} \tau^{j}$.

I. $p$ and $q$ both even.

$$\frac{1}{4}D\varphi \sim \sum B_{j}j(j - \frac{n}{2}) \tau^{j} + \theta(\tau) \sum A_{j}(j + 1)(j - \frac{n}{2} + 1) \tau^{j}.$$

II. $p$ odd and $q$ even.

$$\frac{1}{4}D\varphi \sim \sum B_{j}j(j - \frac{n}{2}) \tau^{j-1} + \theta(\tau) \tau^{j} \sum A_{j}(j + \frac{1}{2})(j - \frac{n}{2} + \frac{1}{2}) \tau^{j-1}.$$  

When both $p$ and $q$ are odd it is directly verified that $D'G = 0$. As 1 is a solution of $D'F = 0$ and 1 and $G$ are linearly independent, the theorem follows.

7. Fourier transforms in $L'$.

Everywhere in the above sections we can (with obvious modifications) change $D$ to $S$ and $D'$ to $S'$. In the following the spaces $H_{s, m}$ and $L'$ refer to $S$. The elements of $H_{s, m}$ then have other properties at infinity.

Define the Fourier transform $F$ and its inverse $F^{-1}$ by
\[(\mathcal{F}f)(\mu) = (2\pi)^{-\frac{1}{2}} \int f(u)e^{-i\mu u} \, du\]

and
\[(\mathcal{F}^{-1}f)(\mu) = (2\pi)^{-\frac{1}{2}} \int f(u)e^{i\mu u} \, du\]

for every \(f \in \mathcal{S}\), where \(u\mu\) is defined on p. 203 above. Define \(\mathcal{F}T\) where \(T \in \mathcal{S}'\) by
\[
\langle \mathcal{F}T, f \rangle = \langle T, \mathcal{F}^{-1}f \rangle \quad \text{for every} \quad f \in \mathcal{S}.
\]

Put \(\mathcal{N}' = \mathcal{N}^{-1} \mathcal{F}\mathcal{N}'\), where \(\mathcal{N}'\) is a linear homeomorphism of \(H'_{s,m}\) onto \(H'_{s,m}\). As \(H'_{s,m}\) is reflexive, it follows that \(\mathcal{N}\) is a linear homeomorphism of \(H_{s,m}\) onto \(H_{s,m}\). Let \(H_{s,m} \ni \phi = Ng\). We have
\[
\langle F, \mathcal{N}\phi \rangle = \langle N'\mathcal{N}'F, g \rangle = \langle \mathcal{F}N'F, g \rangle = \langle F, N\mathcal{F}^{-1}g \rangle
\]
for every \(F \in H'_{s,m}\). This implies that \(\mathcal{N}Ng = N\mathcal{F}^{-1}g\), and we have the following theorem

**Theorem 7.1.** \(\mathcal{F}\) induces a linear homeomorphism \(\mathcal{N}' = \mathcal{N}^{-1} \mathcal{F}\mathcal{N}'\) of \(H'_{s,m}\) onto \(H'_{s,m}\). Its adjoint \(\mathcal{N}\) is a linear homeomorphism of \(H_{s,m}\) onto \(H_{s,m}\) defined by
\[
\mathcal{N} = \{[Ng, N\mathcal{F}^{-1}g] \}.
\]

Furthermore \(\mathcal{N}^2 = 1\).

The last statement follows from \(\mathcal{N}^2Ng = N\mathcal{F}^{-1}\mathcal{F}^{-1}g = Ng\).

We shall now give an explicit expression of the kernel of \(\mathcal{N}\).

Let \(\phi = Ng\). We have
\[
\mathcal{N}\phi(\tau) = N\mathcal{F}^{-1}g(\tau) = \lim_{R \to \infty} \int_{|u| \leq R} \delta(uu - \tau) \, du \int e^{i\mu u} g(\mu) \, d\mu
\]
and
\[
e^{i\mu(\Lambda_\mu)} = \int e^{i\eta u} \delta(\mu\mu - \sigma) \, d\sigma,
\]
where \(\eta(\sigma)\) is a function so that \(\eta\eta = \sigma\), and where \(\Lambda_\mu = \Lambda \in \mathcal{L}\) and \(\Lambda\mu = \eta(\mu\mu)\). Now
\[
\mathcal{N}\phi(\tau) = \lim_{R \to +\infty} \int_{|u| \leq R} \delta(uu - \tau) \left(\int e^{i\mu(A^{-1}u)} \delta(\mu\mu - \sigma) \, d\sigma \right) g(\mu) \, d\mu \, du
\]
\[
= \lim_{R \to +\infty} \left(\int_{|u| \leq R} \delta(uu - \tau) e^{i\mu u} \, du Ng(\sigma) \right) \, d\sigma
\]
\[
= \lim_{R \to +\infty} \int_{R} \Delta(\sigma, \tau) Ng(\sigma) \, d\sigma.
\]
Clearly \(\Delta(\sigma, \tau)\) tends to an element of \(H'_{s,m}(\sigma)\), the element \(\Delta(\sigma, \tau)\) say, as \(R \to \infty\).
(i) If $\sigma = \eta \eta > 0$ we have
\[
\Delta_R(\sigma, \tau) = (2\pi)^{-\frac{1}{2n}} \int_{|u| \leq R} \delta(\omega u - \tau) e^{i \sigma \frac{1}{2} x_1} du
\]
\[
= 2^{-\frac{q}{2}} \Gamma(q+1)^{-1} \int_{|\tau|} \int_{R^2} (q+\tau)^{-\frac{1}{2}} (q-\tau)^{\frac{1}{2}} \cos \theta (\sin \theta)^{\bar{p}} d\theta d\varphi
\]
\[
= 2^{-\frac{q}{2}} \Gamma(q+1)^{-1} \int_{|\tau|} \int_{R^2} (q+\tau)^{-\frac{1}{2}} (q-\tau)^{\frac{1}{2}} J_{\frac{1}{2}}((\sigma(q+\tau))^{\frac{1}{2}}) d\theta d\varphi.
\]
Here $J_\nu$ denotes the Bessel function of order $\nu$. If we put $t = (\sigma(q+\tau))^{\frac{1}{2}}$ and $a(\tau) = (\max(0, \tau))^{\frac{1}{2}}$ we get
\[
(15') \quad \Delta_R(\sigma, \tau) = 2^{1-\frac{q}{2}} \Gamma(q+1)^{-1} \sigma^{-\frac{n}{2}} \int_{a(2\tau)}^{(\sigma(R^2 + t^2))^{\frac{1}{2}}} t^{\bar{p}+1} (t^2 - 2\tau \sigma)^{\frac{1}{2}} J_{\frac{1}{2}}(t) dt.
\]

(ii) If we have $\sigma = \eta \eta < 0$ we get in the same way
\[
(15'') \quad \Delta_R(\sigma, \tau) = 2^{1-\frac{q}{2}} \Gamma(q+1)^{-1} \sigma^{-\frac{n}{2}} \int_{a(2\tau)}^{(-\sigma(R^2 + t^2))^{\frac{1}{2}}} t^{\bar{p}+1} (t^2 - 2\tau \sigma)^{\frac{1}{2}} J_{\frac{1}{2}}(t) dt.
\]
Introduce the notations
\[
c_{p, q}(\sigma) = 2^{1-\frac{q}{2}} \Gamma(q+1)^{-1} \sigma^{-\frac{n}{2}}, \quad \mathcal{H}_{\alpha, \beta}(k) = \int_{a(k)}^{\infty} t^{\frac{1}{2}+1} (t^2 + k)^{-\beta} J_{\alpha}(t) dt.
\]
In order to give $\mathcal{H}_{\alpha, \beta}(k)$ a meaning when $\alpha = \bar{p}$ and $\beta = \bar{q}$, we prove the following lemma

LEMMA 7.1. Let $k$ be fixed.

For fixed $\beta$, $\mathcal{H}_{\alpha, \beta}(k)$ can be continued to a function of $\alpha$ which is analytic in the whole complex plane.

For fixed $\alpha$, $\mathcal{H}_{\alpha, \beta}(k)$ can be continued to a function which is analytic in the whole complex plane if $k < 0$ and analytic in the whole plane except when $\beta = 2, 3, \ldots$, where it has simple poles if $k > 0$.

PROOF. Let $k > 0$. When the integral converges we have
\[
(16) \quad \mathcal{H}_{\alpha, \beta}(k) = \int_{k}^{\infty} t^{\frac{1}{2}+1} (t^2 - k)^{-\beta} J_{\alpha}(t) dt
\]
\[
= \Gamma(1 - \beta) k^{\frac{1}{2}+1 - \beta} \left(e^{\pi \xi} J_{\frac{1}{2} - \beta} - i \sin \pi \xi H_{\frac{1}{2} - \beta}(k^{\frac{1}{2}})\right).
\]
Here $H_{\nu}$ is a Hankel cylinder function which is analytic in $\nu$. Hence the lemma follows for $k > 0$.  

Let $k < 0$. When the integral converges we have

\begin{align}
\mathcal{H}_{\alpha, \beta}(k) &= \int_{0}^{\infty} t^{\alpha+1}(t^2 - k)^{-\beta} J_{\alpha}(t) dt \\
&= i(-k)^{(\alpha - \beta)/2} c(\alpha - \beta) 2^{-\beta-1} \Gamma(\beta+1)^{-1} H_{\alpha-\beta}^1(( -k)^{1/2}).
\end{align}

Hence the lemma follows. — For the formulas (16) and (17) see Nielsen [5, pp. 222–224].

By analytic continuation and from (15′) and (15′′) follows

**Theorem 7.2.** The kernel $\Delta(\sigma, \tau)$ to $\mathcal{N}$ is given by

\[ \mathcal{H}_{p, -\bar{q}}(2\sigma\tau) \text{ Pf. } c_{p, q}(\sigma) \quad \text{when } \sigma > 0 \]

and

\[ \mathcal{H}_{-\bar{p}, q}(2\sigma\tau) \text{ Pf. } c_{q, p}(\sigma) \quad \text{when } \sigma < 0. \]

8. Solutions in $\mathcal{L}'$ of $P(\square)S = T$.

**Lemma 8.1.** If $P$ is a polynomial, the mapping $H_{s, m} \ni \varphi \to P(\tau)\varphi \in H_{s, m}$ has a continuous inverse.

**Proof.** As $P(\tau) = a_0 \prod (\tau - \lambda_j)$ it is sufficient to prove the lemma when $P(\tau) = \tau - \lambda$.

If $\text{Im} \lambda \neq 0$ the lemma is trivial.

Let $\lambda$ be real $\neq 0$. Let $\vartheta_\lambda(\tau) \in \mathcal{D}(R)$ so that $\vartheta_\lambda(\tau) = 1$ for $|\tau| \leq |\lambda|$, and $\vartheta_\lambda = 0$ for $|\tau| > |\lambda|$. Let $\varphi_j(\tau - \lambda)H_{s, m}$ and let $\varphi_j \to 0$ in $H_{s, m}$. Clearly

\[ \varphi_j(\tau) \vartheta_\lambda(\tau)(\tau - \lambda)^{-1} \to 0 \text{ in } H_{s, m}, \]

and as in Schwartz [8, p. 123] it is proved that

\[ \varphi_j(\tau)(1 - \vartheta_\lambda(\tau))(\tau - \lambda)^{-1} \to 0 \text{ in } \mathcal{D}(R). \]

Now we prove the lemma for $\lambda = 0$. Let $\varphi_j \in \tau H_{s, m}$. It is easily seen that supp$(\varphi_j/\tau)$ is contained in a fixed compact set $K \subset R$ and that

\[ A_k(\varphi_j/\tau) \to 0. \]

We have

\[ \sup_{\tau \in K} |D^v_\tau \varphi_j| \to 0, \quad \text{where } \varphi_j(\tau) = \varphi_j(\tau) - \gamma_s(\tau)\chi(\tau)P_\tau(\tau), \]

and

\[ (\varphi_j/\tau) - \sum_{k=0}^{v} A_k(\varphi_j/\tau)\tau^k \gamma_s(\tau)\chi(\tau) = \varphi_j(\tau)/\tau = \int_{0}^{\infty} \varphi_j(\tau \sigma) d\sigma. \]

Now the lemma follows from the inequality

\[ \sup_{\tau \in K} |D^{v-1}_\tau (\varphi_j(\tau)/\tau)| \leq \sup_{\tau \in K} \left| \int_{0}^{1} \varphi_j^{(v)}(\sigma \tau) \sigma^{v-1} d\sigma \right| \leq v^{-1} \sup_{\tau \in K} |D^v_\tau \varphi_j|. \]
Corollary. \( P(\tau)H^\prime_{s,m} = H^\prime_{s,m} \).

Theorem 8.1. The equation \( P(\square)S = T \) with \( T \in \mathcal{L}' \) has a solution in \( \mathcal{L}' \).

Proof. We shall prove that \( P(\square)\mathcal{L}' = \mathcal{L}' \) which clearly is equivalent to proving that \( P(D')H^\prime_{s,m} = H^\prime_{s,m} \). It is well known that \( \mathcal{F} \square \mathcal{F}^{-1} \) is multiplication by \( uu \). Hence \( \mathcal{N}^{-1}D'\mathcal{N}' \) is multiplication by \( \tau \); for if \( H_{s,m} \varphi = Ng \) where \( g \in \mathcal{D}(\mathbb{R}^n) \), then

\[
\langle \mathcal{N}^{-1}D'\mathcal{N}'F, \varphi \rangle = \langle \mathcal{N}'\mathcal{N}^{-1}D'\mathcal{N}'F, g \rangle \\
= \langle \mathcal{F}N'D'\mathcal{N}'F, g \rangle \\
= \langle F, \mathcal{N}N \square \mathcal{F}^{-1}g \rangle \\
= \langle F, N\mathcal{F} \square \mathcal{F}^{-1}g \rangle = \langle \tau F, \varphi \rangle .
\]

Clearly \( P(D')H^\prime_{s,m} = H^\prime_{s,m} \), if and only if \( \mathcal{N}^{-1}P(D')\mathcal{N}'H^\prime_{s,m} = H^\prime_{s,m} \), and as \( \mathcal{N}^{-1}P(D')\mathcal{N}' = P(\mathcal{N}^{-1}D'\mathcal{N}') \) this is true by the corollary of Lemma 8.1.

Remark. This result is also true relative to \( \mathcal{D} \).

References


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