FJELDSTAD'S SERIES
AND THE q-ANALOG OF A SERIES OF DOUGALL

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1.

Fjeldstad [3] proved the identity

\[ \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2n}{n-m+k} \binom{2p}{p-m+k} = (-1)^m \frac{(m+n+p)! (2m)! (2n)! (2p)!}{(m+n)! (m+p)! (n+p)! m! n! p!}, \]

and later Carlitz [1] gave its q-analog

\[ \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2n}{n-m+k} \binom{2p}{p-m+k} q^{k(3(k-m)^2+(k-m))} = (-1)^m \frac{[m+n+p]! [2m]! [2n]! [2p]!}{[m+n]! [m+p]! [n+p]! [m]! [n]! [p]!}, \]

where

\[ \binom{n}{k} = \prod_{j=1}^{k} \frac{q^{n-j+1} - 1}{q^j - 1} = \frac{[n]!}{[k]! [n-k]!}, \]

\[ [n]! = \prod_{j=1}^{n} (q^j - 1), \quad [0]! = 1. \]

It may be of interest to show how (1) follows from a general formula of Dougall [2] and how (2) leads to the q-analog of Dougall’s formula.

2.

As a special case of his more general theorem, Dougall [2, formula (10)] gave the relation

\[ 1 - \frac{2xyz}{(x+1)(y+1)(z+1)} + \frac{2x(x-1)y(y-1)z(z-1)}{(x+1)(x+2)(y+1)(y+2)(z+1)(z+2)} - \ldots = \frac{x! y! z! (x+y+z)!}{(x+y)! (y+z)! (z+x)!}, \quad R(x+y+z) > -1, \]

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which may be rewritten in the form
\[
\sum_{k=0}^{\infty} (-1)^k \binom{x+y+z}{k} \binom{x+y+z+k}{k} = \frac{1}{2} \left\{ \frac{x! \cdot y! \cdot z! \cdot (x+y+z)!}{(x+y)! \cdot (y+z)! \cdot (z+x)!} + 1 \right\}.
\]

We note the binomial coefficient identity
\[
\binom{x+k}{k} \binom{y+k}{k} \binom{z+k}{k} = \binom{x-k}{x} \binom{y-k}{y} \binom{z-k}{z},
\]
and let \( x = n \) be a non-negative integer. Then the relation (4) implies that
\[
\frac{1}{2} \left\{ \frac{(n+y+z)! \cdot (2n)! \cdot (2y)! \cdot (2z)!}{(n+y)! \cdot (n+z)! \cdot (y+z)! \cdot n! \cdot y! \cdot z!} + \frac{1}{2} \binom{2n}{n} \binom{2y}{y} \binom{2z}{z} \right\}
= \sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} \binom{2y}{y-k} \binom{2z}{z-k}
= \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \binom{2y}{y-n+k} \binom{2z}{z-n+k}
= (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2y}{y-n+k} \binom{2z}{z-n+k}
+ \frac{1}{2} \binom{2n}{n} \binom{2y}{y} \binom{2z}{z},
\]
and consequently we have
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2y}{y-n+k} \binom{2z}{z-n+k}
= (-1)^n \frac{(n+y+z)! \cdot (2n)! \cdot (2y)! \cdot (2z)!}{(n+y)! \cdot (n+z)! \cdot (y+z)! \cdot n! \cdot y! \cdot z!}.
\]
When \( y \) and \( z \) are non-negative integers this is equivalent with (1).

3.
To obtain the \( q \)-analog of (4) in the case where \( x = n \) is a non-negative integer we proceed as follows. From (2) we have
\[
\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2n}{n-m+k} \binom{2p}{p-m+k} q^{\frac{1}{2}(k-m)^2+(k-m)}
= \sum_{k=0}^{m} \sum_{k=m}^{2m} (-1)^m \binom{2m}{m} \binom{2n}{n} \binom{2p}{p}
= \sum_{k=0}^{m} (-1)^{m-k} \binom{2m}{m-k} \binom{2n}{n-k} \binom{2p}{p-k} q^{k^2} (q^{2k} + q^{-2k}) - (-1)^m \binom{2m}{m} \binom{2n}{n} \binom{2p}{p},
\]
Now, obviously the identity (5) applies to the \( q \)-binomial coefficients also, and applying this to our equations we find that we have

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n}{k} \binom{p}{k} q^{3k^2}(q^{1k} + q^{-1k})
\]

\[
= 1 + \frac{[m+n+p]! [m]! [n]! [p]!}{[m+n]! [m+p]! [n+p]!},
\]

which is the desired \( q \)-analog of (5) when \( x = m = \) non-negative integer.

4.

An interesting special case of (7) might be noted. Since

\[
\binom{-m-1}{k} = \frac{(-1)^k}{q^{mk} q^{k(k+1)}} \binom{m+k}{k}
\]

it follows that

\[
\frac{\binom{-m-1}{k}}{\binom{-m+k-1}{k}} = \frac{\binom{m+k}{k}}{\binom{m}{k}} q^{-k^2}.
\]

Therefore, setting \( p = -m - 1 \) in (7) we have

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{n+k}{k} (q^k + 1) q^{k(k-1)} = 1 + (-1)^m \binom{n-1}{m} q^{m(m+1)}.
\]

REFERENCES