ON THE MINIMUM MODULUS OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

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1. Introduction.

Much work has been performed to find the connection between the minimum and the maximum modulus of an entire function. Many problems concerning these moduli still remain unsolved. A survey of this field has been given by W. K. Hayman [1].

Let $f(z)$ be an entire function. We denote $\max |f(z)|$ and $\min |f(z)|$ on $|z| = r$ by $M(r)$ and $m(r)$ respectively. The order $\varrho$ and lower order $\lambda$ are defined as $\lim \sup$ and $\lim \inf$ of $\log M(r)/\log r$ as $r \to \infty$.

Many years ago it was proved that

\begin{equation}
\limsup_{r \to \infty} \frac{\log m(r)}{\log M(r)} \geq \cos \pi \varrho
\end{equation}

if $0 < \varrho < 1$, a result later on sharpened by Beurling (cf. [2, pp. 14–16], the result was Theorem II below with $\varrho$ instead of $\lambda$, $0 < \varrho \leq \frac{1}{2}$).

The author has earlier proved (cf. [3]) that $\cos \pi \varrho$ in (1) can be replaced by $\cos \pi \lambda$ if $0 < \lambda < \varrho < 1$. Our purpose now is to remove the condition $\varrho < 1$.

It is the fact that $\log |f(z)|$ is subharmonic which is used in the proof of theorems such as (1). Also in the following the theorems stated for entire functions could be given as theorems for subharmonic functions $u(z)$, replacing $\log |f(z)|$.

2. Three theorems.

**Theorem I.** Let $f(z)$ be an entire function of lower order $\lambda$, $0 < \lambda < 1$. Then

\begin{equation}
\limsup_{r \to \infty} \frac{\log m(r)}{\log M(r)} \geq \cos \pi \lambda.
\end{equation}

**Theorem II.** Let $f(z)$ be an entire function of positive or infinite order. Suppose that

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\[
\liminf_{r \to \infty} \frac{\log M(r)}{r^\lambda} = 0,
\]

\(\lambda\) being a number in \(0 < \lambda < 1\). Then

\[
\log m(r) > \cos \pi \lambda \log M(r)
\]

holds true for certain arbitrarily large values of \(r\).

**Theorem III.** Let \(f(z)\) be an entire function of positive or infinite order. Suppose that

\[
\liminf_{r \to \infty} \frac{\log M(r)}{r^\lambda} < \infty, \quad \text{but} \quad \limsup_{r \to \infty} \frac{\log M(r)}{r^\lambda} = \infty,
\]

\(\lambda\) being a number in \(0 < \lambda < 1\). Then again

\[
\log m(r) > \cos \pi \lambda \log M(r)
\]

holds true for certain arbitrarily large values of \(r\).

Perhaps it should be mentioned here that it is impossible in Theorem I to replace (2) by (4). At the beginning of this century, Wiman [5] constructed functions having \(\lambda = \varrho\) for which (4) does not hold.

Theorem I follows from Theorem II so we have only to prove the last two theorems.

3. **Decomposition of \(f(z)\).**

For the sake of simplicity we first suppose \(f(0) = 1\), a restriction which will be removed later on.

Let us fix a number \(R\), large enough for zeros of \(f(z)\) to exist within \(|z| < R\), these zeros being denoted by \(a_1, a_2, \ldots, a_N\).

Our function \(f(z)\) must have an infinity of zeros. Suppose there were only a finite number, say \(a_1, a_2, \ldots, a_p\). Then

\[
g(z) = \frac{f(z)}{(z-a_1)(z-a_2) \ldots (z-a_p)}
\]

would be an entire function without zeros. Furthermore

\[
|g(z)| < e^{\varepsilon |z|^{\lambda+\varepsilon}}, \quad 0 < \lambda < 1, \quad \varepsilon > 0,
\]

for certain arbitrarily large values of \(|z|\). By a standard argument (cf. Titchmarsh [4, Theorem 8.24]) we then conclude that \(\log g(z)\) must be a polynomial of degree zero and thus \(f(z)\) a polynomial of degree \(p\). This is contrary to the supposition in Theorems I–III, hence there must be an infinity of zeros of \(f(z)\).
We now define
\begin{equation}
 f_1(z) \equiv \prod_{n=1}^{N} (1 - z/\alpha_n) ,
\end{equation}
and
\begin{equation}
 f_2(z) \equiv \prod_{n=1}^{N} (1 + z/|\alpha_n|) .
\end{equation}

A function \( f_3(z) \) is defined by the identity
\begin{equation}
 f(z) \equiv f_1(z)f_3(z) .
\end{equation}

We observe that \( f_3(z) \neq 0 \) for \(|z| < R\).

4. Estimation of \(|f_1(z)|\) and \(|f_2(z)|\).

The maximum and the minimum of \(|f_v(z)|, v = 1, 2, 3\), on \(|z| = r\) are denoted by \( M_v(r) \) and \( m_v(r) \). Obviously
\begin{equation}
 m_2(r) \leq m_1(r) < M_1(r) \leq M_2(r) .
\end{equation}

Let \( n(t) \) be the number of zeros of \( f(z) \) for \(|z| < t\). Jensen's theorem gives
\begin{equation}
 \int_0^{2R} \frac{n(t)}{t} \, dt \leq \log M(2R) \quad \text{thus} \quad n(R) \log 2 \leq \log M(2R) .
\end{equation}

We then estimate \( \log M_2(R) \):
\begin{equation}
 \log M_2(R) = \sum_{1}^{N} \log (1 + R/|\alpha_n|) = \int_0^{R} \log (1 + R/t) \, dn(t)
\end{equation}
\begin{equation}
 = n(R) \log 2 + \int_0^{R} \frac{R}{R+t} \cdot \frac{n(t)}{t} \, dt
\end{equation}
\begin{equation}
 < n(R) \log 2 + \int_0^{R} \frac{n(t)}{t} \, dt
\end{equation}
\begin{equation}
 \leq n(R) \log 2 + \log M(R) < 2 \log M(2R) .
\end{equation}

Hence
\begin{equation}
 \log M_1(R) \leq \log M_2(R) < 2 \log M(2R) .
\end{equation}

5. Estimation of \(|f_3(z)|\).

We obtain from (8)
\begin{equation}
 \log M_3(r) \leq \log M(r) - \log m_1(r) .
\end{equation}

Because \(|\alpha_n| < R\), that is \(|1 - 2R/|\alpha_n|| > 1\), we get
(13) \[ m_1(2R) \geq m_2(2R) = \prod_{1}^{N} \left| 1 - 2R/|a_n| \right| > 1 , \]
and so from (12) \[ \log M_3(2R) < \log M(2R) . \]
Thus

(14) \[ \log M_3(R) < \log M(2R) . \]

Since \( f_3(z) \neq 0 \) for \( |z| < R \) we can define

(15) \[ \psi(z) = \log f_3(z) , \]
where \( \psi(z) \) is regular for \( |z| < R \) and \( \psi(0) = 0 \). By a well-known theorem of Carathéodory (cf. [4, Theorem 5.5]) we get for \( |z| < R \)

(16) \[ |\psi(z)| \leq \frac{2A(R)|z|}{R - |z|} \]

where \( A(R) = \max_{|z|=R} \text{Re}\{\psi(z)\} = \log M_3(R) \). Hence

(17) \[ |\log |f_3(z)|| = |\text{Re}\{\psi(z)\}| \leq |\psi(z)| \leq \frac{2 \log M_3(R)}{R - |z|} \cdot |z| . \]

For \( |z| \leq \frac{1}{4} R \), we get the formula

(18) \[ |\log |f_3(z)|| \leq \frac{4 \log M_3(R)}{R} \cdot |z| . \]

From (14) we at last obtain

(19) \[ |\log |f_3(z)|| \leq \frac{4 \log M(2R)}{R} \cdot |z| , \]
valid for \( |z| \leq \frac{1}{4} R \).

We may observe that, given a function \( f(z) \) of lower order \( \lambda \), \( 0 < \lambda < 1 \), it is possible to choose a sequence \( R_1, R_2, \ldots, R_p, \ldots \to \infty \) such that the right-hand side of (19) tends to zero uniformly in every circle \( |z| \leq R_0 \). Thus \( f(z) \) can be uniformly approximated in \( |z| \leq R_0 \) by means of a sequence of Weierstrassian products of the simplest kind.

6. A formula for \( |f_2(z)| \).

Let \( 0 < R_1 < R_2 \). By taking the integral

(20) \[ \int \frac{\log (1 + z/|a_n|)}{z^{1+\lambda}} \, dz , \]
where \( 0 < \lambda < 1 \), around the upper half of the annulus \( R_1 < |z| < R_2 \), one can deduce (cf. [2, p. 16], and [3, p. 136]) that
\[ \frac{R_2}{R_1} \int_{R_1}^{R_2} \frac{\log |1-r/|a_n|| - \cos \pi \lambda \log (1+r/|a_n|)}{r^{1+\lambda}} \, dr > k(\lambda) \frac{\log (1+R_1/|a_n|)}{R_1^\lambda} - K(\lambda) \frac{\log (1+R_2/|a_n|)}{R_2^\lambda}. \]

The value of \( k(\lambda) \) is (cf. [2, p. 18])

\[ k(\lambda) = \frac{1 - \sin \pi \frac{1}{2} - \lambda}{\frac{1}{2} - |\frac{1}{2} - \lambda|}. \]

The best value of \( K(\lambda) \) seems to be more difficult to obtain, but a rough estimation gives \( K(\lambda) < 10 \).

Summation with respect to \( n \) from 1 to \( N \) gives from (21)

\[ \frac{R_2}{R_1} \int_{R_1}^{R_2} \frac{\log m_n(r) - \cos \pi \lambda \log M_n(r)}{r^{1+\lambda}} \, dr > k(\lambda) \frac{\log M_n(R_1)}{R_1^\lambda} - K(\lambda) \frac{\log M_n(R_2)}{R_2^\lambda}, \]
valid for \( 0 < \lambda < 1, \ 0 < R_1 < R_2 \).

7. Proof of the theorems in the case of \( \lambda \leq \frac{1}{2} \).

Let us add the integral

\[ I(R_1, R_2) = \int_{R_1}^{R_2} \frac{\log m_n(r) - \cos \pi \lambda \log M_n(r)}{r^{1+\lambda}} \, dr \]
to both sides of (23). Denoting

\[ A(R_1, R_2) = \int_{R_1}^{R_2} \frac{\log m_n(r) m_n(r) - \cos \pi \lambda \log M_n(r) M_n(r)}{r^{1+\lambda}} \, dr \]
we then get from (23)

\[ A(R_1, R_2) > k(\lambda) \frac{\log M_n(R_1)}{R_1^\lambda} - K(\lambda) \frac{\log M_n(R_2)}{R_2^\lambda} + I(R_1, R_2). \]

We shall now prove that it is possible to choose arbitrarily large \( R_1 \) and \( R_2 \) such that the right-hand side of (26) is positive. Let us set

\[ \alpha = \lim \inf_{r \to \infty} \frac{\log M(r)}{r^\lambda}, \quad \beta = \lim \sup_{r \to \infty} \frac{\log M(r)}{r^\lambda}. \]

We have supposed in Theorem II that \( \alpha = 0 \), and, in Theorem III, that \( \alpha < \infty \).

We choose a number \( \epsilon > 0 \). For certain arbitrarily large values of \( R \) it then holds that

\[ \log M(2R) < (\alpha + \epsilon)(2R)^\lambda. \]
Let (28) be true for a value $R$. We then choose $R_2 = \frac{1}{2} R$ and estimate the terms in the right-hand side of (26). To begin with (the value of $R_1$ will be chosen later)

\begin{equation}
\log M_2(R_1) = \log M_2(R_1) M_3(R_1) - \log M_3(R_1) \\
\geq \log M(R_1) - \log M_3(R_1).
\end{equation}

From (19) and (28) we get

\begin{equation}
\log M_3(R_1) \leq \frac{4 \log M(2R)}{R} R_1 < 4(\alpha + \epsilon) 2^{\lambda R^1 - 1} R_1 \\
= (\alpha + \epsilon) 2^{\lambda + 2}(R_1/R)^{1 - \lambda} R_1^\lambda.
\end{equation}

Hence

\begin{equation}
\log M_2(R_1) > \log M(R_1) - (\alpha + \epsilon) 2^{\lambda + 2}(R_1/R)^{1 - \lambda} R_1^\lambda.
\end{equation}

Furthermore, from (11) and (28).

\begin{equation}
- \log M_2(R_2) = - \log M_2(\frac{1}{2} R) > - \log M_2(R) \\
> - 2 \log M(2R) > - (\alpha + \epsilon) 2^{\lambda + 1} R_1^\lambda.
\end{equation}

By (19), for $r \leq \frac{1}{2} R$,

\[
\log m_3(r) \geq - \frac{4 \log M(2R)}{R} r, \quad - \log M_3(r) \geq - \frac{4 \log M(2R)}{R} r.
\]

Thus

\begin{equation}
\log m_3(r) - \cos \pi \lambda \log M_3(r) > - \frac{8 \log M(2R)}{R} r > - 8(\alpha + \epsilon) 2^{\lambda R^1 - 1} r.
\end{equation}

Inserting this in (24) we obtain

\[
I(R_1, \frac{1}{2} R) > - (\alpha + \epsilon) 2^{\lambda + 3} R_1^{\lambda - 1} \int_{R_1}^{\frac{1}{2} R} \frac{dr}{r^\lambda} = - \frac{\alpha + \epsilon}{1 - \lambda} 2^{\lambda + 3} R_1^{\lambda - 1} \{(\frac{1}{2} R)^{1 - \lambda} - R_1^{1 - \lambda}\}
\]

and then

\begin{equation}
I(R_1, \frac{1}{2} R) > - \frac{\alpha + \epsilon}{1 - \lambda} 2^{2\lambda + 2}.
\end{equation}

Combining (26), (31), (32) and (34) we get

\begin{equation}
A(R_1, \frac{1}{2} R) > k(\lambda) \frac{\log M(R_1)}{R_1^\lambda} - k(\lambda)(\alpha + \epsilon) 2^{\lambda + 2}(R_1/R)^{1 - \lambda} - \\
- K(\lambda)(\alpha + \epsilon) 2^{\lambda + 2} \frac{\alpha + \epsilon}{1 - \lambda} 2^{2\lambda + 2}.
\end{equation}

Let us now consider the case $\alpha = 0$. For an arbitrarily chosen $R_1$ the first term on the right-hand side of (35) is some positive number. We then choose $\epsilon$ small enough (that is $R$ large) to make the right-hand side of (35) positive.
If $\alpha > 0$ we have supposed that $\beta = \infty$. Then, by appropriate choice of $R_1$, the first term on the right-hand side of (35) can be as large as we wish. Thus there exist also in this case arbitrarily large $R_1$ and $R$ such that $A(R_1, \frac{1}{2}R) > 0$.

Remembering (25) we conclude that

\begin{equation}
\log m_2(r) m_3(r) - \cos \pi \lambda \log M_2(r) M_3(r) > 0
\end{equation}

for certain values of $r \to \infty$. Since $f(z) = f_1(z) f_3(z)$ we have

\begin{equation}
m(r) \geq m_1(r) m_3(r) \geq m_2(r) m_3(r)
\end{equation}

and

\begin{equation}
M(r) \leq M_1(r) M_3(r) \leq M_2(r) M_3(r).
\end{equation}

Because, in this section, we have $\lambda \leq \frac{1}{2}$, that is $\cos \pi \lambda \geq 0$, (36), (37) and (38) yield what is to be proved in Theorems II and III:

\begin{equation}
\log m(r) - \cos \pi \lambda \log M(r) > 0
\end{equation}

for certain values of $r \to \infty$.

8. Proof of the theorems in the case of $\lambda > \frac{1}{2}$.

This time we add the integral

\begin{equation}
J(R_1, R_2) = \int_{R_1}^{R_2} \frac{(1 - \cos \pi \lambda) \log m_3(r)}{r^{1+\lambda}} \, dr
\end{equation}

to both sides of the inequality (23). We perform all the estimations in exactly the same manner as before and finally we conclude that

\begin{equation}
\log m_2(r) m_3(r) - \cos \pi \lambda \log M_2(r) m_3(r) > 0
\end{equation}

for certain values of $r \to \infty$. Because $\cos \pi \lambda < 0$, the inequality $M_2(r) \geq M_1(r)$ now goes in the wrong direction. We avoid this difficulty in the usual manner (cf. [4, Theorem 8.74]): Let $z_0$ be a point on $|z| = r$ where $m_1(r) = |f_1(z_0)|$. Then

\begin{equation}
m_2(r) M_2(r) = \prod_{1}^{N} |1 - r^2/a_n|^2 \leq \prod_{1}^{N} |1 - z_0^2/a_n|^2
\end{equation}

\begin{equation}
= |f_1(z_0)| \cdot |f_1(-z_0)| \leq m_1(r) M_1(r).
\end{equation}

Hence, by means of (40),

\begin{equation}
m_1(r) M_1(r) \geq m_2(r) M_2(r) > M_2(r)^{1+\cos \pi \lambda} m_3(r) \cos \pi \lambda - 1
\end{equation}

\begin{equation}
\geq M_1(r)^{1+\cos \pi \lambda} m_3(r) \cos \pi \lambda - 1,
\end{equation}

which gives

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\[ m_1(r) m_3(r) > M_1(r)^{\cos \pi \lambda} m_3(r)^{\cos \pi \lambda} \]

or
\[ \log m_1(r) m_3(r) - \cos \pi \lambda \log M_1(r) m_3(r) > 0. \]

Since \( m(r) \geq m_1(r) m_3(r) \) and \( M(r) \geq M_1(r) m_3(r) \), we obtain, once again, in the case \( \frac{1}{2} < \lambda < 1 \) that
\[ \log m(r) - \cos \pi \lambda \log M(r) > 0 \]

for certain arbitrarily large values of \( r \).

9. Zeros at the origin.

For simplicity we have up to now supposed \( f(0) = 1 \). If there are \( p \) zeros at the origin, the entire function can be written
\[ f(z) = A z^p f_1(z) f_3(z). \]

There \( f_1(z) \) is defined as in (6) and \( f_3(z) \) by (44) and by \( f_3(0) = 1 \). To complete the proof in order to cover also this case, we must in addition add to (23) the integral
\[ \int_{R_1}^{R_2} \frac{(1 - \cos \pi \lambda)(\log |A| + p \log r)}{r^{1+\lambda}} \, dr. \]

The small changes in the proof which are necessary do not affect the conclusions about the positive sign of the right-hand sides of the formulae corresponding to (35). Thus (4) still holds, and Theorems I–III are proved without any restriction on \( f(0) \).

In Theorem III, the condition \( \beta = \infty \) is stronger than necessary. As we can see from (35), \( \beta \) should be larger than a certain multiple of \( \alpha \):
\[ \beta > \frac{2^{1+2\lambda}}{k(\lambda)} \left( K(\lambda) + \frac{2}{1-\lambda} \right) \alpha, \]

but, of course, the value of the factor is far from being the best. But if \( \beta < \infty \), the function is of order \( \lambda \), \( 0 < \lambda < 1 \), a case not of much interest here.

REFERENCES
2. B. Kjellberg, On certain integral and harmonic functions, Thesis, Uppsala, 1948. (Copies will be sent upon request to Prof. B. Kjellberg, Royal Institute of Technology, Stockholm 70.)
3. B. Kjellberg, A relation between the maximum and minimum modulus of a class of entire
functions, C. R. du 12. Congrès des Mathématiciens Scandinaves tenu à Lund 10–15 août 1953, Lund, 1954, 135–138. (The exponent in the right-hand side of formula (11) on p. 137 is incorrect, it should be \( \lambda - \varepsilon \), not \( \sigma - \varepsilon \).)


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