PYTHAGOREAN INEQUALITIES FOR CONVEX BODIES

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Consider a convex body in Euclidean three dimensional space and let \( \sigma(u) \) be the area of the projection of this body upon a plane with normal direction \( u \). It was conjectured by C. Carathéodory and proved by W. Blaschke [1, p. 148] that if \( u_1, u_2, u_3 \) are three mutually perpendicular directions, then

\[
\sigma^2(u) \leq \sum \sigma^2(u_i).
\]

In this note a class of such Pythagorean inequalities for convex bodies, (which includes that of Carathéodory), is shown to follow from one particular inequality. Also, in a certain sense, the cases of equality are determined.

Let \( K \) be a convex body in Euclidean \( n \)-dimensional space \( (n \geq 2) \), \( B(u) \) the breadth of \( K \) in the direction \( u \). That is, if \( h(u) \) is the support function of \( K \), then

\[
B(u) = h(u) + h(-u).
\]

Suppose \( \{u_i\}, i = 1, 2, \ldots, n \), are a set of \( n \) mutually perpendicular directions and consider the circumscribing rectangular parallelepiped of \( K \) which is bounded by the \( 2n \) supporting hyperplanes of \( K \) having outer normal directions \( \pm u_i \). Call this circumscribing body \( C \). Finally, let \( P(u) \) be some point common to \( K \) and its supporting hyperplane with outer normal direction \( u \). For any \( u \), the points \( P(u) \) and \( P(-u) \) belong to the body \( C \) and so the distance \( d \) between them is less than or equal to the length of a diagonal of \( C \), that is \( d^2 \leq \sum B^2(u_i) \). On the other hand, since \( B(u) \) is the minimum distance between points of the hyperplanes containing \( P(u) \) and \( P(-u) \), we have

\[
B^2(u) \leq \sum B^2(u_i).
\]

The class of Pythagorean inequalities which we have in mind follow from (1). If \( V(K_1, \ldots, K_{n-1}, u) \) denotes the mixed volume of convex bodies \( K_1, \ldots, K_{n-1} \) and the unit segment in the direction \( u \), then \( V(K_1, \ldots, K_{n-1}, u) \) as a function of \( u \), is known to be the support function of a convex body having a centre of symmetry [2, p. 44]. The breadth

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of this body in the direction \( u \) is then \( 2V(K_1, \ldots, K_{n-1}, u) \) and so, with the preceding meaning for \( u_i \), we have

\[
V^2(K_1, \ldots, K_{n-1}, u) \leq \sum V^2(K_1, \ldots, K_{n-1}, u_i).
\]

In particular, if we let \( K_1 = K_2 = \ldots = K_p = K \) and \( K_{p+1} = \ldots = K_{n-1} = S \) (where \( S \) denotes the unit spherical body), then

\[
V(K_1, \ldots, K_{n-1}, u) = W'_{n-p-1}(K, u)
\]

which is the \((n-p-1)\)th cross-section integral of the projection of \( K \) upon a hyperplane with normal direction \( u \) [2, p. 49]. In particular, \( W'_0(K, u) = \sigma(u) \) for which (2) is Carathéodory's inequality.

Returning to (1), we introduce a Pythagorean defect \( \delta(K) \) of \( K \) by

\[
\delta(K) = \max_{(u_i)} \left( \min_u \frac{\sum B^2(u_i) - B^2(u)}{B^2(u)} \right)
\]

\[
= \max_{(u_i)} \left( \frac{\sum B^2(u_i)}{\max_u B^2(u)} - 1 \right).
\]

We have

**Theorem.** \( 0 \leq \delta(K) \leq n - 1 \). Both bounds are attained, the lower being attained if and only if \( K \) is a segment.

Inequality (1) asserts that \( \delta \) is non-negative. By Pythagoras' theorem, \( \delta = 0 \) if \( K \) is a segment. On the other hand, if \( \delta = 0 \), from (3)

\[
\sum B^2(u_i) = \max_u B^2(u)
\]

for all \( \{u_i\} \). Let \( \bar{u}_1 \) be a direction in which \( B(u) \) attains its maximum and let \( \bar{u}_2, \ldots, \bar{u}_n \) be such that \( \{\bar{u}_i\} \) form a mutually perpendicular set of \( n \) directions. Then \( \sum_{i+1} B^2(\bar{u}_i) = 0 \), whence \( B^2(\bar{u}_j) = 0 \) for \( j = 2, 3, \ldots, n \) and so \( K \) is a one dimensional convex body, that is, a segment.

To show \( \delta \leq n - 1 \) we need only remark that \( B^2(u_i)/\max_u B^2(u) \leq 1 \) with equality if and only if \( B^2(u_i) = \max_u B^2(u) \). Therefore, if and only if \( K \) admits a circumscribing hypercube of edge length equal to the maximum breadth of \( K \), we have \( \delta = n - 1 \). This is the case, for example, if \( K \) is of constant breadth, or if \( K \) is an \( n \)-dimensional hypercube.

We finally note that the cases of equality in (2) can be found directly by suitably interpreting the statement of the theorem. Thus, for each \( \{u_i\} \) there is a \( u \) for which we have equality in Carathéodory's inequality if and only if \( K \) lies in a hyperplane. On the other hand, the ratio \( \sum \sigma^2(u_i)/\sigma^2(u) \) attains its maximum of \( n \) for a body of constant brightness or for a hypercube.
REFERENCES


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