SOME ARITHMETIC SUMS CONNECTED WITH THE GREATEST INTEGER FUNCTION

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1.

Jacobsthal [3] has introduced the sum

\[ S(a, b, m; r) = \sum_{k=0}^{r-1} D(k), \]

where

\[ D(k) = D(a, b, m; k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor; \]

here \( a, b \) are arbitrary integers while \( m \geq 1, r \geq 1 \). Jacobsthal proved the inequality

\[ S(a, b, m; r) \geq 0. \]

The writer [1] has given another proof of (3) making use of the representation

\[ S(a, b, m; r) = -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(\zeta^{sa} - 1)(\zeta^{sb} - 1)(\zeta^{sr} - 1)}{(\zeta^s - 1)(\zeta^s - 1)}, \]

where \( \zeta = e^{2\pi i/m} \). If we put

\[ J(a, b, c) = -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(\zeta^{sa} - 1)(\zeta^{sb} - 1)(\zeta^{sc} - 1)}{(\zeta^s - 1)(\zeta^s - 1)} \]

for arbitrary \( a, b, c \), then clearly \( J(a, b, c) \) is symmetric in \( a, b, c \); also it is evident that

\[ S(a, b, m; r) = J(a, b, r). \]

It follows easily from (5) that

\[ J(a, b, c) = J(-a, -b, -c). \]

Since \( J(a, b, c) \) has period \( m \) in each variable and

\[ J(a, b, c) = 0 \quad (abc = 0), \]

we may assume that

\[ 1 \leq a \leq m-1, \quad 1 \leq b \leq m-1, \quad 1 \leq c \leq m-1. \]

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Moreover in view of the symmetry and (6) we may also assume that
\begin{equation}
(8) \quad b \leq m - a \leq c
\end{equation}
and
\begin{equation}
(9) \quad b + c \leq m .
\end{equation}
It is proved in [1] that when (7), (8), (9) hold, then
\begin{equation}
(10) \quad J(a, b, c) = b .
\end{equation}
We recall that the Bernoulli function $\bar{B}_p(x)$ is defined for $0 \leq x \leq 1$ by
\[ \bar{B}_p(x) = B_p(x), \quad \bar{B}_p(x + 1) = \bar{B}_p(x) , \]
where $B_p(x)$, the Bernoulli polynomial of degree $p$, is defined by
\[ \frac{t e^{tx}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(x) \frac{t^p}{p!} . \]
Now put
\begin{equation}
(11) \quad D_p(k) = D_p(a, b, m; k)
\end{equation}
\[ = - \bar{B}_p \left( \frac{a+b+k}{m} \right) + \bar{B}_p \left( \frac{a+k}{m} \right) + \bar{B}_p \left( \frac{b+k}{m} \right) - \bar{B}_p \left( \frac{k}{m} \right) . \]
\begin{equation}
(12) \quad S_p(a; b, m; r) = \sum_{k=0}^{r-1} D_p(k) .
\end{equation}
Making use of the formula [2, p. 521]
\[ \bar{B}_p \left( \frac{r}{m} \right) = \frac{B_p}{m^p} + \frac{p}{m^p} \sum_{s=0}^{m-1} \frac{\zeta^{-rs}}{\zeta^s - 1} H_{p-1}(\zeta^{-s}) , \]
which holds for all integral $r$ and $p \geq 1$, we get the representation
\begin{equation}
(13) \quad S_p(a, b, m; r) = - \frac{p}{m^p} \sum_{s=1}^{m-1} H_{p-1}(\zeta^{-s}) \frac{(\zeta^{-as} - 1)(\zeta^{-bs} - 1)(\zeta^{-rs} - 1)}{(\zeta^s - 1)(\zeta^{-s} - 1)} ,
\end{equation}
where $H_n(\lambda)$ is defined by
\[ \frac{1 - \lambda}{e^\lambda - 1} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{t^n}{n!} \quad (\lambda \neq 1) . \]
If we put
\begin{equation}
(14) \quad J_p(a, b, c) = - \frac{p}{m^p} \sum_{s=1}^{m-1} H_{p-1}(\zeta^{-s}) \frac{(\zeta^{-as} - 1)(\zeta^{-bs} - 1)(\zeta^{-cs} - 1)}{(\zeta^s - 1)(\zeta^{-s} - 1)}
\end{equation}
then $J_p(a, b, c)$ has period $m$ in each variable and is symmetric in $a, b, c$; also
\[ S_p(a, b, m; r) = J_p(a, b, r) . \]

Comparing (14) with (5) it is clear that
\[ J(a, b, c) = J_1(a, b, c) . \]

2.

It is by no means evident how to extend (10) to the case of arbitrary \( p \geq 1 \), or, in particular, to frame a theorem that will reduce to (3) when \( p = 1 \). In the present note we limit ourselves to the special cases \( p = 2 \) and 3. It is easily verified that
\[ H_1(\lambda) = (\lambda - 1)^{-1} ; \]
thus (14) becomes
\[ J_2(a, b, c) = -\frac{2}{m^2} \sum_{s=1}^{m-1} \frac{(c - a s - 1)(c - b s - 1)(c - c s - 1)}{(c - s - 1)(c - s - 1)^2} . \]

It follows easily from (15) and (5) that
\[ J_2(a, b, c) + J_2(-a, -b, -c) = -\frac{2}{m} J_1(a, b, c) . \]

As above there is no loss in generality in assuming that (7), (8), (9) hold. We rewrite (15) as
\[ -\frac{1}{2} m^2 J_2(a, b, c) + (m - a)bc \]
\[ = \sum_{s=0}^{m-1} (1 + \zeta^s + \ldots + \zeta^{(m-a-1)s})(1 + \zeta^{-s} + \ldots + \zeta^{-(b-1)s})(1 + \zeta^{-(c-1)s}) \]
and apply the familiar formula
\[ \sum_{s=0}^{m-1} \zeta^{rs} = \begin{cases} m & (m \mid r) \\ 0 & (m \nmid r) \end{cases} . \]

Making use of (7), (8) and (9) we get
\[ -\frac{m^2}{2} J_2(a, b, c) + (m - a)bc = m \sum_{i=0}^{m-a-1} \sum_{j=0}^{b-1} \sum_{k=0}^{c-1} \frac{1}{i+j+k} \]
\[ = m \sum_{i=0}^{b-1} \sum_{j+k=i} 1 + m \sum_{i=b}^{m-a-1} \sum_{j+k=i} \frac{1}{j+k} \]
\[ = m \sum_{i=0}^{b-1} (i-1) + m \sum_{i=b}^{m-a-1} \frac{b}{i} \]
\[ = \frac{1}{2} mb(b+1) + mb(m-a-b) \]
\[ = mb(m-a) - \frac{1}{2} mb(b-1) . \]
It therefore follows without much trouble that
\[
\frac{1}{2}m^2J_2(a, b, c) = -(m-a)(m-c)b + \frac{1}{2}mb(b-1).
\]
Combining (18) with (16) and using (10) we get also
\[
\frac{1}{2}m^2J_2(-a, -b, -c) = (m-a)(m-c)b - \frac{1}{2}mb(b+1).
\]
Thus by means of (18) and (19), \(J_2\) is evaluated for all \(a, b, c\), the notation being such that (7), (8), (9) are satisfied. In particular note that (18) and (19) imply
\[
-(m-a)(m-c)b \leq \frac{1}{2}m^2J_2(a, b, c) \leq \frac{1}{2}mb(b-1),
\]
\[
-\frac{1}{2}mb(b-1) \leq \frac{1}{2}m^2J_2(-a, -b, -c) \leq (m-a)(m-c)b;
\]
these inequalities may be compared with (3).

It is also clear from (18) and (19) that \(\frac{1}{2}m^2J_2\) is integral; indeed we have \(\frac{1}{2}m^2J_2(a, b, c) \equiv -abc \pmod{m}\), \(\frac{1}{2}m^2J_2(-a, -b, -c) \equiv abc \pmod{m}\).

3.

For \(p = 3\), since
\[
H_3(\lambda) = \frac{\lambda + 1}{(\lambda - 1)^2},
\]
we find that
\[
J_3(a, b, c) = -\frac{3}{m^2} \sum_{s=1}^{m-1} (\zeta^{-as} + 1) \frac{(\zeta^{-bs} - 1)(\zeta^{-cs} - 1)}{(\zeta^{-s} - 1)(\zeta^{-s} - 1)^3}.
\]
Then
\[
\frac{1}{2}m^2J_3(a, b, c) = \frac{1}{2}m^2J_2(a, b, c) - 2K,
\]
where
\[
K = \sum_{s=1}^{m-1} \frac{(\zeta^{-as} - 1)(\zeta^{-bs} - 1)(\zeta^{-cs} - 1)}{(\zeta^{-s} - 1)(\zeta^{-s} - 1)^3}.
\]
Now it follows from the identity
\[
\sum_{s=1}^{m-1} \frac{\zeta^{-rs}}{\zeta^{-s} - x} = m\zeta^{r-1} - \frac{1}{1-x^m} - \frac{1}{1-x} \quad (1 \leq r \leq m)
\]
that
\[
\sum_{s=1}^{m-1} \frac{\zeta^{-rs}}{\zeta^{-s} - 1} = \frac{1}{2}(m+1) - r \quad (1 \leq r \leq m).
\]
Rewrite (24) as
\[
K = \sum_{s=1}^{m-1} \frac{1}{\zeta^{s} - 1} (1 + \zeta^{s} + \ldots + \zeta^{(m-a-1)s}) \cdot (1 + \zeta^{-s} + \ldots + \zeta^{-(b-1)s})(1 + \zeta^{-s} + \ldots + \zeta^{-(c-1)s}).
\]
Then by (25) we get

\[ K = \sum_{i=0}^{m-a-1} \sum_{j=0}^{b-1} \sum_{k=0}^{c-1} \left\{ \frac{1}{2}(m+1) - R(j+k-i) \right\}, \]

where \( R(k) \) is defined by

\[ R(k) \equiv k \pmod{m}, \quad 1 \leq R(k) \leq m. \]

Next, assuming that (7), (8) and (9) are satisfied, we have

\[ \sum_{i=0}^{m-a-1} \sum_{j=0}^{b-1} \sum_{k=0}^{c-1} R(j+k-i) = \sum_{i,j,k} (j+k-i) + \sum_{i,j,k} (m+j+k-i) \]

\[ = \sum_{i,j,k} (j+k-i) + m \sum_{i,j,k} 1 = S_1 + mS_2, \]

say. Clearly

\[ S_1 = -\frac{1}{2}(m-a)(m-a-1)bc + \frac{1}{2}(m-a)bc(b-1) + \frac{1}{2}(m-a)bc(c-1) \]

\[ = \frac{1}{2}(m-a)bc(a+b+c-m-1), \]

while

\[ S_2 = \sum_{i=0}^{b-1} \sum_{j=i}^{m-a-1} \sum_{k=i-j}^{1} 1 + \sum_{i=0}^{m-a-1} \sum_{j=i}^{b-1} \sum_{k=i-j}^{b-1} 1 \]

\[ = \sum_{i=0}^{b-1} \sum_{j=i}^{m-a-1} (i-j+1) + \sum_{i=0}^{m-a-1} \sum_{j=i}^{b-1} (i-j+1) \]

\[ = \sum_{i=0}^{b-1} \frac{1}{2}(i+1)(i+2) + \sum_{i=b}^{m-a-1} \{(i+1)b - \frac{1}{2}b(b-1)\} \]

\[ = \frac{1}{2}b(b+1)(b+2) + \frac{1}{2}b(m-a)(m-a+1) \]

\[ - \frac{1}{2}b^2(b+1) - \frac{1}{2}b(b-1)(m-a-b) \]

\[ = \frac{1}{2}b(b-1)(b-2) + \frac{1}{2}b(m-a)(m-a-b+2). \]

Since by (26)

\[ K = \frac{1}{2}(m+1)(m-a)bc - S_1 - mS_2, \]

a little manipulation now yields

\[ K = (m+1)(m-a)bc - \frac{1}{2}(m-a)(a+b+c)bc - \frac{1}{2}m(m-a)bc \cdot \left( m-a-b+2 \right) \]

\[ \cdot (m-a-b+2)b - \frac{1}{2}mb(b-1)(b-2). \]

We may rewrite (27) as

\[ 2K = (m-a)b(m-c)(a+b+c-m-2) - \frac{1}{2}mb(b-1)(b-2). \]

Finally, using (18), (23), we get
(28) \( \frac{1}{3} m^3 J_3(a, b, c) \) 
\[ = -b(m-a)(m-c)(a+b+c-m-1) + \frac{1}{3} mb(b-1)(2b-1) . \]

Since, from (22),
\[ m^3 J_3(-a, -b, -c) = m^3 J_3(a, b, c) + 3m^2 J_2(a, b, c) + 3m J_1(a, b, c) , \]

it follows that

(29) \( \frac{1}{3} m^3 J_3(-a, -b, -c) \)
\[ = -b(m-a)(m-c)(a+b+c-m+1) + \frac{1}{3} mb(b+1)(2b+1) . \]

It is evident that (28) and (29) imply the inequalities

(30) 
\[ -b(m-a)(m-c)(a+b+c-m-1) \leq \frac{1}{3} m^3 J_3(a, b, c) \leq \frac{1}{3} mb(b-1)(2b-1) , \]

(31) 
\[ -b(m-a)(m-c)(a+b+c-m+1) \leq \frac{1}{3} m^3 J_3(-a, -b, -c) \leq \frac{1}{3} mb(b+1)(2b+1) . \]

These inequalities may be compared with (20) and (21).

It is clear from (28) and (29) that \( \frac{1}{3} m^3 J_3 \) is integral; indeed we have

\( \frac{1}{3} m^3 J_3(a, b, c) \equiv -abc(a+b+c-1) \pmod{m} , \)
\[ \frac{1}{3} m^3 J_3(-a, -b, -c) \equiv -abc(a+b+c+1) \pmod{m} . \]

The evaluation of \( J_p \) by the above method for \( p > 3 \) becomes very cumbersome.

REFERENCES


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