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A TYPE OF SERIALLY BALANCED EXPERIMENTAL DESIGNS

ELSE SANDVED

1. Introduction

When a number of different treatments are applied to one or several experimental subjects in successive periods, models of the following type have been considered: Each experimental result is a sum of the direct treatment effect, residual effects of treatments applied in earlier periods, effect of the experimental subject, effect of the period or a common block effect for some neighbouring periods, plus a stochastic term (error) ε which may or may not be autocorrelated. The a priori assumption is that all ε's have expectations equal to zero, $E(\varepsilon)=0$, and common variance equal to σ^2, $\operatorname{var}(\varepsilon)=\sigma^2$.

Finney [2] has considered five models involving combinations of error correlations, residual effects of treatments applied in earlier periods, and autoregressive effects (i.e., response of the subject on one occasion affects its capacity to respond on the next). We shall only consider the case where the model contains no autoregressive effects and the errors are uncorrelated.

The usual procedure in this case is to estimate the unknown parameters by the least squares method, i.e., to minimize

$$Q = \sum \left(x - E(x)\right)^2$$

with respect to the parameters; here, the x's are the experimental responses and the $E(x)$'s are functions of the parameters appearing in the model.

When the model is given, the least squares equations indicate what balancing arrangements are required to make the statistical analysis of the experiment relatively simple. Balanced designs also make the estimates more efficient than estimates based on unbalanced designs with the same amount of experimental material.

Williams [7] [8] considers designs where each treatment is applied once to each experimental subject in successive periods. He discusses the construction and analysis of designs balanced with respect to the residual

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effect of the treatment applied in the previous period, i.e. the first residual effect, and the residual effect of the treatments applied in the two previous periods. His designs are partly based on latin squares and sets of latin squares.

When the first residual effects only are to be estimated, a simple example has been described by Cochran, Autrey and Cannon [1] for 3 different treatments applied in 3 successive periods to each experimental subject.

Designs where the number of treatments is greater than the number of periods for each experimental subject consist of incomplete sequences and have been investigated by Patterson [5]. He has indicated 7 balance conditions for the case where first residual effects are estimated and has given several rules for constructing balanced designs, especially under the restriction that the number of experimental subjects is to be as small as possible.

In [4] he has treated the analysis of the designs.

When v treatments are to be applied to a single experimental subject in successive periods, each treatment has to be applied several times to this subject. This case has been considered by Finney and Outhwaite [3]. They consider the model

$$x_{ijk} = \mu + b_i + t_j + r_k + \varepsilon_{ijk} ,$$

$$\sum_{i=1}^{b} b_i = \sum_{j=1}^{v} t_j = \sum_{k=1}^{v} r_k = 0 ,$$

where x_{ijk} is the experimental response in block number i in the latter of two subsequent periods in which treatments k and j, respectively, were applied. We assume μ, b_i, t_j and r_k to be unknown constants, while the ε_{ijk} are random variables, all independent and normally distributed with zero mean and common variance σ^2. The parameters are interpreted as follows: μ is the general mean, b_i is the block effect, t_j is the direct treatment effect, and r_k is the residual effect from the treatment in the preceding period. The b_i's, t_j's and r_k's are measured as deviations from the general mean.

The least squares equations assume a convenient form if we base the experimental design on a type 1 sequence or a type 2 sequence of the kind introduced by Finney and Outhwaite [3]:

A *type* 1 *sequence of index k and order v* involves an arrangement of v different symbols such that

(i) the sequence consists of (kv^2+1) symbols, amongst which one occurs $(kv+1)$ times and the others kv times each;

(ii) after the first symbol, the remaining kv successive sets of length v form blocks of complete replicates;

(iii) every one of the v^2 possible ordered pairs of successive symbols occurs k times.

An example with index 2 and order 3 is

1 123 312 231 132 213 321

Balanced sequences of small index k will be of practical importance.

For index $k = 1$, several balanced sequences have been constructed. For $v = 2$ there exists a trivial sequence. For $v = 3$, 4, and 5, enumeration has established that no sequences exist. For $v = 6$ Anne D. Outhwaite has discovered a number of distinct sequences. M. R. Sampford [6] has, by different methods, found a number of sequences for different values of v; there is at least one sequence for each value of v up to 11. Further, he has discovered a method for finding a sequence with $v = 4r + 2$, $r = 1$, 2, 3, ..., when a sequence with $v = 2r + 1$ is known.

In section 2, I shall present a general method for constructing a type 1 sequence of index 1 and order $v = 4r + 2$, $r = 1, 2, 3, \ldots$

For index $k = 2$, Sampford has given a method for constructing balanced type 1 sequences of general order v.

A *type* 2 *sequence of index* k *and order* v involves an arrangement of v different symbols such that

(i) the sequence consists of $(kv^2 - kv + 1)$ symbols, amongst which one occurs $(kv - k + 1)$ times and the others $k(v - 1)$ times each;

(ii) after the first symbol, the remaining $k(v - 1)$ successive sets of length v form blocks of complete replicates;

(iii) every one of the $v(v - 1)$ possible ordered pairs of successive *different* symbols occurs k times.

An example with index 2 and order 3 is

1 213 123 132 321

For index $k = 1$, Sampford gives a method for constructing type 2 sequences of general order v.

In both types of sequences the first observation is omitted from the analysis.

When each block in a type 2 sequence has the same symbol at the end, and the sequence accordingly is totally reversible, every residual effect occurs in each block. This is a great advantage in analyzing the experimental results. An example of a totally reversible balanced sequence of type 2 with index 1 and order 5 is

1 24531 52341 35421 43251

Sampford has discovered methods for constructing totally reversible type 2 sequences of index $k = 1$ and order $v = 2r + 1$ and of index $k = 2$ and general order v.

If we write down n balanced sequences of the same type, of index k and order v, one after the other, omitting all initial elements except the first one, the result will be a balanced sequence of the same type as the original one, of index nk and order v.

Sampford has also treated the analysis of serially balanced sequences.

2. A method for constructing a type 1 sequence of index $k = 1$ and order $v = 4r+2$, $r = 1, 2, 3, \ldots$

We shall represent the $v=4r+2$ symbols by unstarred and starred residues modulo $2r+1$: $0, 1, 2, \ldots, 2r, 0^*, 1^*, 2^*, \ldots, (2r)^*$. The balanced sequence is formed of residues $n_{x,y}$ and $n^*_{x,y}$ in a $(4r+2)\times(4r+2)$ square, where x denotes the number of the column and y denotes the number of the row $(x, y=1, 2, \ldots, 4r+2)$.

(1) We first put the unstarred residues $n_{x,y}$ in the upper half of the square $(y=1, 2, \ldots, 2r+1)$ in the following way:

On the line	$n_{x,y} =$	y is	$y =$
$x = y-2r$	r	odd	$2r+1$
$x = y-2(r-1)$	$r-1$	odd	$2(r-1)+1,\ 2r+1$
	$r+2$	even	$2(r-1)+2$
.....	.....	.....	.....
$x = y-2k$	k	odd	$2k+1, 2k+2, \ldots, 2r+1$
	$2r+1-k$	even	
.....	.....	.....	.....
$x = y-4$	2	odd	$5, 6, \ldots, 2r+1$
	$2r-1$	even	
$x = y-2$	1	odd	$3, 4, \ldots, 2r+1$
	$2r$	even	
$x = y$	0		$1, 2, \ldots, 2r+1$
$x = y+2$	$2r$	odd	$1, 2, \ldots, 2r+1$
	1	even	
$x = y+4$	$2r-1$	odd	$1, 2, \ldots, 2r+1$
	2	even	
.....	.....	.....	.....
$x = y+2k$	$2r+1-k$	odd	$1, 2, \ldots, 2r+1$
	k	even	
.....	.....	.....	.....
$x = y+2r$	$r+1$	odd	$1, 2, \ldots, 2r+1$
	r	even	
$x+y = 3$	1	odd	$1, 2$
	$2r$	even	

On the line	$n_{x,y} =$	y is	$y =$
$x+y = 5$	$\begin{cases} 2 \\ 2r-1 \end{cases}$	$\begin{matrix} \text{odd} \\ \text{even} \end{matrix}\Big\}$	1, 2, 3, 4
....	....	....	....
$x+y = 2k+1$	$\begin{cases} k \\ 2r+1-k \end{cases}$	$\begin{matrix} \text{odd} \\ \text{even} \end{matrix}\Big\}$	$1, 2, \ldots, 2k$
....	....	....	....
$x+y = 2r+1$	$\begin{cases} r \\ r+1 \end{cases}$	$\begin{matrix} \text{odd} \\ \text{even} \end{matrix}\Big\}$	$1, 2, \ldots, 2r$

(2) Now, in one half of the upper half of the square we have put unstarred residues. In the rest of the upper half of the square we put starred residues according to the formula

$$n^*_{x,y} = (n_{4r+3-x,y}+1)^* .$$

(3) In the lower half of the square the symbols are placed symmetrically with respect to the horizontal midline of the square, except that the unstarred and starred residues are interchanged.

Example with $r = 4$.

$y \diagdown x$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	8	2	7	3	6	4	5	6*	5*	7*	4*	8*	3*	0*	2*	1*
2	8	0	7	1	6	2	5	3	5*	4	4*	6*	3*	7*	2*	8*	1*	0*
3	1	2	0	3	8	4	7	6*	6	7*	5	8*	5*	0*	4*	1*	3*	2*
4	7	8	6	0	5	1	5*	2	4*	3	3*	4	2*	6*	1*	7*	0*	8*
5	2	3	1	4	0	6*	8	7*	7	8*	6	0*	5	1*	5*	2*	4*	3*
6	6	7	5	8	5*	0	4*	1	3*	2	2*	3	1*	4	0*	6*	8*	7*
7	3	4	2	6*	1	7*	0	8*	8	0*	7	1*	6	2*	5	3*	5*	4*
8	5	6	5*	7	4*	8	3*	0	2*	1	1*	2	0*	3	8*	4	7*	6*
9	4	6*	3	7*	2	8*	1	0*	0	1*	8	2*	7	3*	6	4*	5	5*
10	4*	6	3*	7	2*	8	1*	0	0*	1	8*	2	7*	3	6*	4	5*	5
11	5*	6*	5	7*	4	8*	3	0*	2	1*	1	2*	0	3*	8	4*	7	6
12	3*	4*	2*	6	1*	7	0*	8	8*	0	7*	1	6*	2	5*	3	5	4
13	6*	7*	5*	8*	5	0*	4	1*	3	2*	2	3*	1	4*	0	6	8	7
14	2*	3*	1*	4*	0*	6	8*	7	7*	8	6*	0	5*	1	5	2	4	3
15	7*	8*	6*	0*	5*	1*	5	2*	4	3*	3	4*	2	6	1	7	0	8
16	1*	2*	0*	3*	8*	4*	7*	6	6*	7	5*	8	5	0	4	1	3	2
17	8*	0*	7*	1*	6*	2*	5*	3*	5	4*	4	6	3	7	2	8	1	0
18	0*	1*	8*	2*	7*	3*	6*	4*	5*	6	5	7	4	8	3	0	2	1

The required sequence is then formed by linking together all $4r+2$ rows in such a way that, if the residue a is the first residue of a row in the sequence, then the next row in the sequence starts with the residue of $(a+1)^*$, and if a^* is the first residue in a row in the sequence, then the next row in the sequence starts with the residue of $(a+1)$. We may form for instance the sequence

$$0(0 \ldots 1^*)\ (1^* \ldots 2)\ (2 \ldots 3^*) \ldots$$
$$\ldots (2r \ldots 0^*)\ (0^* \ldots 1)\ (1 \ldots 2^*) \ldots \big((2r)^* \ldots 0\big).$$

Theorems 1–3, to be stated and proved below, show that this is a type 1 sequence of index 1 and order $4r+2$. To prove the theorems, we first make, by means of (1) and (2), a survey of the coordinates of an unstarred residue a and of a starred residue b^* in the upper half of the square:

$$0 \leqq a \leqq r.$$

a occurs in

$$\begin{aligned}
(x,y) &= (1+2g,\ 2a+1+2g), && g = 0,1,\ldots,r-a, \\
(x,y) &= (2a+2j,\ 2j), && j = 1,2,\ldots,r, \\
(x,y) &= (2a-2k,\ 1+2k), && k = 0,1,\ldots,a-1.
\end{aligned} \tag{4}$$

$$r+1 \leqq a \leqq 2r.$$

a occurs in

$$\begin{aligned}
(x,y) &= (2n,\ 4r-2a+2+2n), && n = 1,2,\ldots,a-r-1, \\
(x,y) &= (4r-2a+3+2p,\ 1+2p), && p = 0,1,\ldots,r, \\
(x,y) &= (4r-2a+1-2q,\ 2+2q), && q = 0,1,\ldots,2r-a.
\end{aligned} \tag{5}$$

$$1^* \leqq b^* \leqq (r+1)^*.$$

b^* occurs in

$$\begin{aligned}
(x,y) &= (4r+2-2g^*,\ 2b-1+2g^*), && g^* = 0,1,\ldots,r-b+1 \\
(x,y) &= (4r-2b+5-2j^*,\ 2j^*), && j^* = 1,2,\ldots,r \\
(x,y) &= (4r-2b+5+2k^*,\ 1+2k^*), && k^* = 0,1,\ldots,b-2
\end{aligned} \tag{6}$$

$$(r+2)^* \leqq b^* \leqq (2r+1)^*.$$

b^* occurs in

$$\begin{aligned}
(x,y) &= (4r+3-2n^*,\ 4r-2b+4+2n^*), && n^* = 1,2,\ldots,b-r-2 \\
(x,y) &= (2b-2-2p^*,\ 1+2p^*), && p^* = 0,1,\ldots,r \\
(x,y) &= (2b+2q^*,\ 2+2q^*), && q^* = 0,1,\ldots,2r-b+1
\end{aligned} \tag{7}$$

In (7) and throughout the proof we shall use $(2r+1)^*$ in the calculation instead of 0^*, and we shall write $b=2r+1$ when $b^*=(2r+1)^*$.

THEOREM 1. *When a square is constructed according to* (1), (2) *and* (3), *every row contains each of the* $4r+2$ *different symbols.*

Proof. It follows from (4) and (5) that the $2r+1$ different unstarred residues occur in each of the rows $y=1, 2, \ldots, 2r+1$. The theorem then follows from (2) and (3).

Theorem 2. *When a square is constructed according to* (1), (2) *and* (3), *the* $4r+2$ *symbols with the coordinate* $x=1$ *are all different.*

Proof. It follows from (4) and (5) that each of the $2r+1$ different unstarred residues occurs in the column $x=1$ in the upper half of the square. The theorem then follows from (3).

Theorem 3. *When a square is constructed according to* (1), (2) *and* (3), *every ordered pair of two different symbols occurs exactly once in the rows of the square.*

Proof. The proof is very tedious. During the proof we apply the following calculation rules for residues modulo $2r+1$:

If $a \neq b$:

$$\mathrm{res}(b-a) \textit{ odd} \longleftrightarrow \mathrm{res}(a-b) \textit{ even}$$

If $a<b$:

$$\mathrm{res}(b-a) \textit{ odd} \longleftrightarrow a+b \textit{ odd} \longleftrightarrow \begin{cases} \mathrm{res}(a+b) \textit{ odd if } a+b \leqq 2r, \\ \mathrm{res}(a+b) \textit{ even if } a+b > 2r. \end{cases}$$

$$\mathrm{res}(b-a) \textit{ even} \longleftrightarrow a+b \textit{ even} \longleftrightarrow \begin{cases} \mathrm{res}(a+b) \textit{ even if } a+b \leqq 2r, \\ \mathrm{res}(a+b) \textit{ odd if } a+b > 2r. \end{cases}$$

We are first going to prove that *every ordered pair of unstarred residues* (a, b), *where* $a \neq b$ *and* $\mathrm{res}(b-a)$ *is odd, occurs in the rows of the upper half of the square.*

When this is proved it follows from the construction that every ordered pair of starred residues $c^* d^*$, where $c^* \neq d^*$ and $\mathrm{res}(d^*-c^*)$ is even, occurs in the rows of the upper half of the square. Further, the rows of the lower half of the square will contain every ordered pair ab, where $a \neq b$ and $\mathrm{res}(b-a)$ is even, and every ordered pair $c^* d^*$, where $c^* \neq d^*$ and $\mathrm{res}(d^*-c^*)$ is odd.

Next we are going to prove that *every ordered pair* $a\, b^*$ *where* $\mathrm{res}(a+b)=1, 2, 4, 6, \ldots, 2r$, *and every ordered pair* $b^*\, a$ *where* $\mathrm{res}(a+b)= 0, 3, 5, 7, \ldots, 2r-1$, *occurs in the rows of the upper half of the square.*

When this is proved, it follows from the construction that every ordered pair $a^*\, b$, where $\mathrm{res}(a+b)=1, 2, 4, 6, \ldots, 2r$ and every ordered pair $b\, a^*$, where $\mathrm{res}(a+b)=0, 3, 5, 7, \ldots, 2r-1$ occurs in the rows of the lower half of the square.

We shall distinguish various types of ordered pairs $a\, b$; $a\, b^*$; $b^*\, a$.

$$1^\circ \quad 0 \leqq a \leqq r, \quad 0 \leqq b \leqq r\,.$$

a) $a\,b$; *$a<b$ and* $\operatorname{res}(b-a)$ *is odd.* It then follows from the first and third formula (4) that $a\,b$ occurs where

$$\begin{aligned} 2a+1+2g &= 1+2k, & g &= 0, 1, \ldots, r-a\,,\\ 1+2g+1 &= 2b-2k, & k &= 0, 1, \ldots, b-1\,. \end{aligned}$$

Hence we get in fact

$$\begin{aligned} 0 &\leqq g = \tfrac12(b-a-1) \leqq \tfrac12(r-a)-\tfrac12 < r-a\,,\\ 0 &\leqq k = \tfrac12(a+b-1) \leqq b-1\,, \end{aligned}$$

that is, g and k are integers within their admissible ranges.

b) $a\,b$; *$a>b$ and* $\operatorname{res}(b-a)$ *is odd.* We get from the first and third formula (4)

$$\begin{aligned} 1+2k &= 2b+1+2g, & k &= 0, 1, \ldots, a-1\,,\\ 2a-2k+1 &= 1+2g, & g &= 0, 1, \ldots, r-b\,, \end{aligned}$$

Hence

$$\begin{aligned} 0 &< k = \tfrac12(a+b) \leqq a-1\,,\\ 0 &< g = \tfrac12(a-b) \leqq r-b; \end{aligned}$$

the values of k and g are admissible.

$$2^\circ \quad 0 \leqq a \leqq r, \quad r+1 \leqq b \leqq 2r\,.$$

a) $a\,b$; *$a+b \leqq 2r-1$ and* $\operatorname{res}(b-a)$ *is odd.* It follows from the second formula (4) and the third formula (5) that $a\,b$ occurs where

$$\begin{aligned} 2j &= 2+2q, & j &= 1, 2, \ldots, r\,,\\ 2a+2j+1 &= 4r-2b+1-2q, & q &= 0, 1, \ldots, 2r-b\,. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leqq q = \tfrac12\big(2r-1-(a+b)\big) \leqq \tfrac12(2r-b)-\tfrac12 < 2r-b\,,\\ 1 &\leqq j = 1+q \leqq \tfrac12\big(2r-1-(r+1)\big)+1 < r; \end{aligned}$$

the values of q and j are admissible.

b) $a\,b$; *$a+b \geqq 2r+1$ and* $\operatorname{res}(b-a)$ *is odd.* We get from the third formula (4) and the second formula (5)

$$\begin{aligned} 1+2k &= 1+2p, & k &= 0, 1. \ldots, a-1\,,\\ 2a-2k+1 &= 4r-2b+3+2p, & p &= 0, 1, \ldots, r\,. \end{aligned}$$

Hence

$$0 \leqq p = k = \tfrac12(a+b-2r-1) \leqq \tfrac12(a+2r-2r-1) \leqq a-1\,,$$

as $a \geqq 1$ because $a+b \geqq 2r+1$. The values of k and p are admissible.

$$3^\circ \quad r+1 \leqq a \leqq 2r, \quad 0 \leqq b \leqq r\,.$$

a) $a\,b$; $a+b \leqq 2r$ *and* $\operatorname{res}(b-a)$ *is odd.* We get from the third formula (5) and the second formula (4)

$$\begin{aligned} 2+2q &= 2j, & q &= 0, 1, \ldots, 2r-a\,, \\ 4r-2a+1-2q+1 &= 2b+2j, & j &= 1, 2, \ldots, r\,. \end{aligned}$$

Hence

$$\begin{aligned} 0 \leqq q &= \tfrac{1}{2}\big(2r-(a+b)\big) \leqq \tfrac{1}{2}(2r-a) \leqq 2r-a\,, \\ 1 \leqq j &= q+1 = r-\tfrac{1}{2}(a+b)+1 \leqq \tfrac{1}{2}(r+1) \leqq r; \end{aligned}$$

the values of q and j are admissible.

b) $a\,b$; $a+b \geqq 2r+2$ *and* $\operatorname{res}(b-a)$ *is odd.* We get from the second formula (5) and the third formula (4)

$$\begin{aligned} 1+2p &= 1+2k, & p &= 0, 1, \ldots, r\,, \\ 4r-2a+3+2p+1 &= 2b-2k, & k &= 0, 1, \ldots, b-1\,. \end{aligned}$$

Hence

$$0 \leqq p = k = \tfrac{1}{2}(a+b-2r-2) \leqq \tfrac{1}{2}(2r+b-2r-2) = \tfrac{1}{2}b-1 \leqq b-1 < r;$$

the values of p and k are admissible.

$$4^\circ \quad r+1 \leqq a \leqq 2r, \quad r+1 \leqq b \leqq 2r\,.$$

a) $a\,b$; $a<b$ *and* $\operatorname{res}(b-a)$ *is odd.* We get from the third and first formula (5)

$$\begin{aligned} 2+2q &= 4r-2b+2+2n, & q &= 0, 1, \ldots, 2r-a\,, \\ 4r-2a+1-2q+1 &= 2n, & n &= 1, 2, \ldots, b-r-1\,. \end{aligned}$$

Hence

$$\begin{aligned} 0 < q &= 2r-\tfrac{1}{2}(a+b)+\tfrac{1}{2} \leqq 2r-\tfrac{1}{2}(a+a+1)+\tfrac{1}{2} = 2r-a\,, \\ 1 \leqq n &= \tfrac{1}{2}(b-a+1) = \tfrac{1}{2}(b-a-1)+1 \leqq b-a-1+1 \leqq b-r-1; \end{aligned}$$

the values of q and n are admissible.

b) $a\,b$; $a>b$ *and* $\operatorname{res}(b-a)$ *is odd.* We get from the first and third formula (5)

$$\begin{aligned} 4r-2a+2+2n &= 2+2q, & n &= 1, 2, \ldots, a-r-1\,, \\ 2n+1 &= 4r-2b+1-2q, & q &= 0, 1, \ldots, 2r-b\,. \end{aligned}$$

Hence

$$\begin{aligned} 0 < q &= 2r-\tfrac{1}{2}(a+b) < 2r-b\,, \\ 1 \leqq n &= \tfrac{1}{2}(a-b) \leqq \tfrac{1}{2}(a-r-1) \leqq a-r-1; \end{aligned}$$

the values of q and n are admissible.

So far we have proved that every ordered pair $a\,b$, where $\operatorname{res}(b-a)$ *is odd, occurs in the rows of the upper half of the square.*

We next consider the ordered pairs $a\,b^*$ and $b^*\,a$ occurring in the rows of the upper half of the square.

$$5^\circ \quad 0 \leqq a \leqq r, \quad 1^* \leqq b^* \leqq (r+1)^* .$$

a) $a\,b^*$; $\operatorname{res}(a+b)=2, 4, \ldots, 2r$. We get from the second formula (4) and the second formula (6)

$$\begin{aligned} 2j &= 2j^*, & j &= 1, 2, \ldots, r , \\ 2a+2j+1 &= 4r-2b+5-2j^*, & j^* &= 1, 2, \ldots, r . \end{aligned}$$

Hence

$$1 \leqq j = j^* = r-\tfrac{1}{2}(a+b)+1 \leqq r;$$

the values of j and j^* are admissible.

If $\operatorname{res}(a+b)=1$, then $a=0$ and $b=1$, and we get a solution from the first formula (4) and the first formula (6) for $g=g^*=r$.

b) $b^*\,a$; $\operatorname{res}(a+b)=3, 5, \ldots, 2r-1$. We get from the second formula (6) and the second formula (4)

$$\begin{aligned} 2j^* &= 2j, & j^* &= 1, 2, \ldots, r , \\ 4r-2b+5-2j^*+1 &= 2a+2j, & j &= 1, 2, \ldots, r . \end{aligned}$$

Hence

$$1 < j = j^* = r+\tfrac{3}{2}-\tfrac{1}{2}(a+b) \leqq r;$$

the values of j and j^* are admissible.

If $\operatorname{res}(a+b)=0$, then $a=r$ and $b=(r+1)$, and we get a solution from the second formula (6) and the second formula (4) for $j=j^*=1$.

$$6^\circ \quad 0 \leqq a \leqq r, \quad (r+2)^* \leqq b^* \leqq (2r+1)^* .$$

a) $a\,b^*$; *$a+b$ is an even number less than or equal to* $2r+2$. We get from the first formula (4) and the second formula (7)

$$\begin{aligned} 2a+1+2g &= 1+2p^*, & g &= 0, 1, \ldots, r-a , \\ 1+2g+1 &= 2b-2-2p^*, & p^* &= 0, 1, \ldots, r . \end{aligned}$$

Hence

$$\begin{aligned} &0 \leqq g = \tfrac{1}{2}(b-a)-1 = \tfrac{1}{2}(a+b)-a-1 \leqq r-a , \\ &0 < \tfrac{1}{2}r \leqq p^* = \tfrac{1}{2}(a+b)-1 \leqq r; \end{aligned}$$

the values of g and p^* are admissible.

b) $a\,b^*$; *$a+b$ is an odd number greater than or equal to* $2r+3$. We get from the second formula (4) and the first formula (7)

$$\begin{aligned} 2j &= 4r-2b+4+2n^*, & j &= 1, 2, \ldots, r , \\ 2a+2j+1 &= 4r+3-2n^*, & n^* &= 1, 2, \ldots, b-r-2 . \end{aligned}$$

Hence

$$1 < \tfrac{1}{2}r+1 \leqq j = 2r+\tfrac{3}{2}-\tfrac{1}{2}(a+b) \leqq r\,,$$
$$1 \leqq n^* = \tfrac{1}{2}(b-a-1) = b-\tfrac{1}{2}(a+b+1) \leqq b-r-2;$$

the values of j and n^* are admissible. But $a+b$ even and $\leqq 2r+2$, and $a+b$ odd and $\geqq 2r+3$, correspond to $\operatorname{res}(a+b) = 1, 2, 4, 6, \ldots, 2r$.

c) $b^*\,a$; *$a+b$ is an odd number less than or equal to $2r+1$.* We get from the second formula (7) and the first formula (4)

$$1+2p^* = 2a+1+2g, \qquad p^* = 0, 1, \ldots, r\,,$$
$$2b-2-2p^*+1 = 1+2g, \qquad g = 0, 1, \ldots, r-a\,.$$

Hence

$$0 < \tfrac{1}{2}(r+1) \leqq p^* = \tfrac{1}{2}(a+b-1) \leqq r\,,$$
$$0 < g = \tfrac{1}{2}(b-a-1) = \tfrac{1}{2}(b+a-1)-a \leqq r-a;$$

the values of p^* and g are admissible.

d) $b^*\,a$; *$a+b$ is an even number greater than or equal to $2r+4$.* We get from the first formula (7) and the second formula (4)

$$4r-2b+4+2n^* = 2j, \qquad n^* = 1, 2, \ldots, b-r-2\,,$$
$$4r+3-2n^*+1 = 2a+2j, \qquad j = 1, 2, \ldots, r\,.$$

Hence

$$1 \leqq n^* = \tfrac{1}{2}(b-a) = b-\tfrac{1}{2}(a+b) \leqq b-r-2\,,$$
$$1 < \tfrac{1}{2}(r+3) \leqq j = 2r+2-\tfrac{1}{2}(a+b) \leqq r;$$

the values of n^* and j are admissible. But $a+b$ odd and $\leqq 2r+1$, and $a+b$ even and $\geqq 2r+4$, correspond to $\operatorname{res}(a+b) = 0, 3, 5, \ldots, 2r-1$.

$$7^\circ \quad r+1 \leqq a \leqq 2r, \quad 1^* \leqq b^* \leqq (r+1)^*\,.$$

a) $\operatorname{res}(a+b) = 1, 2, 4, 6, \ldots, 2r$. As a consequence of the construction, the ordered pair $a\,b^*$ occurs in the rows of the upper half of the square if and only if the ordered pair $((b-1), (a+1)^*)$, where $0 \leqq b-1 \leqq r$ and $(r+2)^* \leqq (a+1)^* \leqq (2r+1)^*$, occurs in the rows of the upper half of the square, and we have just proved that the latter ordered pair occurs in the rows of the upper half of the square.

b) $\operatorname{res}(a+b) = 0, 3, 5, 7, \ldots, 2r-1$. As a consequence of the construction, the ordered pair $b^*\,a$ occurs in the rows of the upper half of the square if and only if the ordered pair $((a+1)^*, (b-1))$, where $(r+2)^* \leqq (a+1)^* \leqq (2r+1)^*$ and $0 \leqq b-1 \leqq r$, occurs in the rows of the upper half of the square, and we have just proved that the latter ordered pair occurs in the rows of the upper half of the square.

$$8^\circ \quad r+1 \leqq a \leqq 2r, \quad (r+2)^* \leqq b^* \leqq (2r+1)^*\,.$$

As $2r+3 \leqq a+b \leqq 4r+1$, $\operatorname{res}(a+b)$ cannot be equal to 0 or 1.

a) $a\,b^*$; *$a+b$ is an odd number.* This corresponds to $\operatorname{res}(a+b) = 2, 4, \ldots, 2r$. We get from the second formula (5) and the second formula (7)

$$\begin{aligned} 1+2p &= 1+2p^*, & p &= 0, 1, \ldots, r\,, \\ 4r-2a+3+2p+1 &= 2b-2-2p^*, & p^* &= 0, 1, \ldots, r\,. \end{aligned}$$

Hence

$$0 \leqq p = p^* = \tfrac{1}{2}(a+b-2r-3) < r;$$

the values of p and p^* are admissible.

b) $b^*\,a$; *$a+b$ is an even number.* This corresponds to $\operatorname{res}(a+b) = 3, 5, \ldots, 2r-1$.

We get from the second formula (7) and the second formula (5)

$$\begin{aligned} 1+2p^* &= 1+2p, & p^* &= 0, 1, \ldots, r\,, \\ 2b-2-2p^*+1 &= 4r-2a+3+2p, & p &= 0, 1, \ldots, r\,. \end{aligned}$$

Hence

$$0 < 1 \leqq p = p^* = \tfrac{1}{2}(a+b)-r-1 \leqq r-1 < r;$$

the values of p and p^* are admissible.

This concludes the proof of Theorem 3.

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UNIVERSITY OF OSLO, NORWAY