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ON THE EQUIVALENCE BETWEEN PROXIMITY STRUCTURES AND TOTALLY BOUNDED UNIFORM STRUCTURES

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In the present paper we point out how the proximity structures of V. A. Efremovič [4] [5] [1], can be treated within the framework of A. Weil's uniform structures [6] [2], and how Yu. M. Smirnov's theorems on compactification of proximity spaces [5] may be traced back to well known properties of uniform spaces. Similar problems are discussed in a paper by I. S. Gál [7] which has been published since the present paper was submitted.

1. Two theorems on uniform neighbourhood systems. The first theorem is essentially a restatement of a result of Yu. M. Smirnov, asserting that there exists a coarsest uniform structure compatible with a given proximity structure [5, p. 565]. However, we offer a new proof, which is based directly on A. Weil's original system of axioms [6] [2, Ch. II, p. 131], and not on the alternative system applied by Yu. M. Smirnov [5, p. 563]. (In an appendix to his article Yu. M. Smirnov gives an equivalence proof which he ascribes to A. Kočetkov [5, p. 572].) We state and prove the theorem as a proposition concerning uniform neighbourhood systems. The proximity aspects are postponed to section 2.

Let S be a set provided with a uniform structure $\mathscr{U}$. If A and B are two subsets of S for which there exists an *entourage* V, such that $V(A) \subset B$, then we shall say that B is a *uniform neighbourhood* of A (w.r.t. to $\mathscr{U}$), and we shall write $A \Subset B\,(\mathscr{U})$, or simply $A \Subset B$. A finite covering $\{A_i\}_{1 \leqq i \leqq n}$ of some $A \subset S$ will be called a *p-covering* of A, if there exists another covering $\{B_i\}_{1 \leqq i \leqq n}$ of A such that $B_i \Subset A_i$, $i = 1, \ldots, n$. (The notation $A \Subset B$, and also the term "p-covering" are originally introduced by Yu. M. Smirnov in connection with proximity structures [5, p. 548] [5, p. 559]. The letter "δ" used as a prefix to denote "proximity" is here consequently replaced by the letter "p".)

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THEOREM 1. *For every uniform structure on S we have:*

(P 1) $S \Subset S$

(P 2) $A \Subset B \Rightarrow A \subset B$

(P 3) $A \Subset B \subset C \Rightarrow A \Subset C$

(P 4) $A \Subset B_i,\ i = 1, \ldots, n \Rightarrow A \Subset \bigcap_{i=1}^n B_i$

(P 5) $A \Subset B \Rightarrow \complement B \Subset \complement A$

(P 6) $A \Subset B \Rightarrow \exists C : A \Subset C \Subset B$

Conversely, if $A \Subset B$ is some binary relation between subsets of S satisfying (P 1)–(P 6), *then there exists a uniform structure $\mathcal{U}$ on S, such that*

$$A \Subset B \Leftrightarrow B\,(\mathcal{U})\,. \tag{1.1}$$

In particular, there exists a coarsest uniform structure, $\mathcal{U}_\omega$, with this property; and a fundamental system of entourages of $\mathcal{U}_\omega$ consists of the sets

$$V = \bigcup_{i=1}^{n} (A_i \times A_i)\,, \tag{1.2}$$

where $\{A_i\}_{1 \leqq i \leqq n}$ is a p-covering of S.

Here the notion of "p-covering" is tacitly transferred to the case in which the relation $A \Subset B$ is not *a priori* derived from any uniform structure.

PROOF. 1° The statements (P 1)–(P 6) are easily verified.

2° We assume (P 1)–(P 6) satisfied, and we first list some of their immediate consequences which will be needed in the sequel:

$$A \subset B \Subset C \Rightarrow A \Subset C\,; \tag{1.3}$$

$$\left.\begin{array}{c} A_i \Subset B_i \\ i = 1, \ldots, n \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \bigcap_{i=1}^{n} A_i \Subset \bigcap_{i=1}^{n} B_i\,, \\ \bigcup_{i=1}^{n} A_i \Subset \bigcup_{i=1}^{n} B_i\,. \end{array}\right. \tag{1.4}$$

If $\{A_i\}_{1 \leqq i \leqq n}$ is a p-covering of S, we choose the covering $\{B_i\}_{1 \leqq i \leqq n}$ such that $B_i \Subset A_i$, $i = 1, \ldots, n$. Then for $1 \leqq r \leqq n$ we have

$$\complement \bigcup_{i=1}^{r} A_i \Subset \complement \bigcup_{i=1}^{r} B_i \subset \bigcup_{i=r+1}^{n} B_i \Subset \bigcup_{i=r+1}^{n} A_i\,,$$

proving

$$\complement \bigcup_{i=1}^{r} A_i \Subset \bigcup_{i=r+1}^{n} A_i, \qquad 1 \leqq r \leqq n\,. \tag{1.5}$$

Now we pass to the proof that the collection $\mathscr{V}$ of sets of the form (1.2) is a fundamental system of *entourages* of a uniform structure. The only difficulty consists in verifying that to each $V \in \mathscr{V}$ there exists a $W \in \mathscr{V}$, such that

$$\overset{2}{W} \subset V.$$

Let V be of the form (1.2) and write $A_i' = \bigcup_{j \neq i} A_j$. Then V is expressible as a finite intersection $V = \bigcap_{i=1}^n V_i$ where

$$V_i = (A_i \times A_i) \cup (A_i' \times A_i') , \tag{1.6}$$

$\{A_i, A_i'\}$ p-covering of S. For each i, $i = 1, \ldots, n$, repeated use of (P 6) enable us to obtain auxilliary sets $C_{i,1}$, $C_{i,2}$, $C_{i,3}$, $C_{i,4}$, such that

$$\complement A_i' \Subset C_{i,1} \Subset C_{i,2} \Subset C_{i,3} \Subset C_{i,4} \Subset A_i .$$

Defining

$$B_i = C_{i,2}, \qquad B_i' = C_{i,4} \cap \complement C_{i,1}, \qquad B_i'' = \complement C_{i,3}$$

we obtain

$$B_i \cap B_i'' = \emptyset, \qquad B_i \cup B_i' \subset A_i, \qquad B_i' \cup B_i'' \subset A_i' . \tag{1.7}$$

Moreover, $\{B_i, B_i', B_i''\}$ is a p-covering of S for each $i, i = 1, \ldots, n$. Hence the sets

$$W_i = (B_i \times B_i) \cup (B_i' \times B_i') \cup (B_i'' \times B_i'') ,$$

have the form (1.2), and in virtue of (1.6) and (1.7) we have

$$\overset{2}{W_i} \subset V_i, \qquad i = 1, 2, \ldots, n .$$

We now define $W = \bigcap_{i=1}^n W_i$. Then W is of the form (1.2) as well as W_i, and we have

$$\overset{2}{W} \subset \bigcap_{i=1}^{n} \overset{2}{W_i} \subset \bigcap_{i=1}^{n} V_i = V, \qquad \text{q.e.d.}$$

3° We shall prove that the uniform structure defined by (1.2) satisfies (1.1). First let $A \Subset B$. Then $\{B, \complement A\}$ is a p-covering af S, and writing

$$V = (B \times B) \cup (\complement A \times \complement A) ,$$

we obtain $V(A) = B$, proving $A \Subset B$ $(\mathscr{U}_\omega)$.

To prove the converse, we have to verify that the given relation satisfies $A \Subset V(A)$ for any $A \subset S$ and any V of the form (1.2). Let A_i, $i = 1, \ldots, n$, have the same meaning as in (1.2), and suppose the enumeration performed such that $A \cap A_i = \emptyset$ for $1 \leqq i \leqq r$, $A \cap A_i \neq \emptyset$ for $r < i \leqq n$, where $1 \leqq r \leqq n$. Then we have

$$A \subset \complement \bigcup_{i=1}^{r} A_i, \qquad V(A) = \bigcup_{i=r+1}^{n} A_i ,$$

which by (1.5) gives $A \Subset V(A)$, q.e.d.

4° Let $\mathcal{U}$ be some uniform structure on S satisfying (1.1). We shall prove that each V of the form (1.2) must be an *entourage* of $\mathcal{U}$. As V is of the form (1.2), we may find a covering $\{B_i\}_{1 \leqq i \leqq n}$ of S such that $B_i \Subset A_i$, $i = 1, \ldots, n$, and then there exist *entourages* W_i of $\mathcal{U}$ such that $W_i(B_i) \subset A_i$. Defining $W = \bigcap_{i=1}^n W_i$, we obtain $W(B_i) \subset A_i$, $i = 1, \ldots, n$. As $\{B_i\}_{1 \leqq i \leqq n}$ is a covering of S, this gives

$$W \subset \bigcup_{i=1}^{n} \left(W(B_i) \times W(B_i)\right) \subset V ,$$

proving $V \in \mathcal{U}$, q.e.d.

A uniform structure $\mathcal{U}$ on S will be said to be *totally bounded* if the entire space S is a totally bounded set with respect to $\mathcal{U}$, i.e. if it is possible for every *entourage* V of $\mathcal{U}$ to find a finite covering, $\{A_i\}_{1 \leqq i \leqq n}$, of S such that $A_i \times A_i \subset V$, $i = 1, \ldots, n$.

THEOREM 2. *If* $A \Subset B$ *is some binary relation between subsets of* S *satisfying* (P 1)–(P 6), *then the uniform structure*, $\mathcal{U}_\omega$, *defined by* (1.2) *is the unique totally bounded structure for which* (1.1) *is valid.*

PROOF. 1° $\mathcal{U}_\omega$ is evidently totally bounded, since for every *entourage* V of the form (1.2), the corresponding covering $\{A_i\}_{1 \leqq i \leqq n}$ of S satisfies the requirement $A_i \times A_i \subset V$, $i = 1, \ldots, n$.

2° To prove the uniqueness, we assume $\mathcal{U}$ to be some totally bounded uniform structure satisfying (1.1), and we shall verify that $\mathcal{U}$ is coarser than $\mathcal{U}_\omega$. For a given *entourage* W of $\mathcal{U}$ we determine the *entourage* W_0 of $\mathcal{U}$ such that

$$\overset{3}{W_0} \subset W.$$

Let $\{A_i\}_{1 \leqq i \leqq n}$ be a covering of S for which $A_i \times A_i \subset W_0$, $i = 1, \ldots, n$. Then $\{W_0(A_i)\}_{1 \leqq i \leqq n}$ is a p-covering of S, and we have

$$\bigcup_{i=1}^{n} \left(W_0(A_i) \times W_0(A_i)\right) \subset \overset{3}{W_0} \subset W , \tag{1.8}$$

which proves W to be an *entourage* of $\mathcal{U}_\omega$ as well, q.e.d.

2. Equivalence with proximity structures. In this section we shall apply the results of section 1 to sketch the equivalence between proximity structures and totally bounded uniform structures.

The term "proximity structure" shall be applied also to "non separating" structures. (Thus, in the axiomatic system of [5, p. 546] we shall substitute $E3'$ in place of $E3$).

The notation $A \Subset B$ is originally defined for proximity spaces by

Yu. M. Smirnov [5, p. 548], (and may be pronounced "B is a p-neighbourhood of A"). In terms of the basic proximity relation, $A\,\delta B$, of V. A. Efremovič [4], or rather its negation, $A\,\bar{\delta} B$, the definition takes the form:

$$A \Subset B \Leftrightarrow A\,\bar{\delta}\,\complement B\,. \tag{2.1}$$

Yu. M. Smirnov proves that the relation $A \Subset B$, thus defined, has the properties (P 1)–(P 6) [5, p. 548]. (Actually (P 1) is not listed among the others, but is easily seen to follow directly from the axiom Б4). Conversely, one may prove without difficulty that a relation $A \Subset B$ satisfying (P 1)–(P 6), via (2.1) defines a proximity structure. Thus, we may take (P 1)–(P 6) as an alternative system of axioms of proximity spaces.

Now the results of section 1 inform us that *the requirement* (1.1) *determines a* 1–1 *correspondence between the totally bounded uniform structures on* S *and the proximity structures on* S. In particular the separation axiom of proximity spaces (i.e. Б3 of [5, p. 546]) may be written

$$\bigcap_{\{x\} \Subset A} A = \{x\}\,.$$

Hence *a proximity structure is separating if and only if the corresponding totally bounded uniform structure is separating.*

We recall that a mapping f from a proximity space into another is called *p-continuous* if it preserves proximity i.e. if $A\,\delta B \Rightarrow f(A)\,\delta\,f(B)$, or what is equivalent, if

$$A \Subset B \Rightarrow f^{-1}(A) \Subset f^{-1}(B)\,. \tag{2.2}$$

(B. A. Efremovič uses this concept without giving it a name [4, p. 190], while Yu. M. Smirnov uses a term which may be rendered in English by "p-mapping" ("*δ-отображение*") [5, p. 550].)

If f is a uniformly continuous mapping of a uniform space into another, then it easily verified that f satisfies (2.2) (the relations $A \Subset B$ being defined in terms of uniform neighbourhoods). Conversely, for a p-continuous function f defined between proximity spaces, f^{-1} preserves p-coverings, and thus $(f\times f)^{-1}$ preserves *entourages* of the form (1.2). Hence *p-continuity is nothing but uniform continuity with respect to the corresponding totally bounded uniform structures.*

We now pass to the compactification of proximity spaces. We recall that a proximity structure has an *associated topology*, for which the neighbourhoods of x are the sets A, such that $\{x\} \Subset A$. (This of course is also the associated topology of every uniform structure satisfying (1.1), and in particular of $\mathscr{U}_{\omega}$.)

In a uniform space $V(A) \subset B \Rightarrow \bar{A} \subset \mathring{B}$, and hence we have the general implication $A \Subset B \Rightarrow \bar{A} \subset \mathring{B}$. In a compact space (where the uniform structure is uniquely determined by the topology [2, ch, II, p. 157]), $\bar{A} \subset \mathring{B}$ implies the existence of an *entourage* V such that $V(A) \subset B$. Hence $\bar{A} \subset \mathring{B} \Rightarrow A \Subset B$ in this case. *Thus, there is a unique proximity structure defining a compact topology, and it is determined by the formula*

$$A \Subset B \Leftrightarrow \bar{A} \subset \mathring{B}\,. \tag{2.3}$$

By a well-known theorem, the completion $\hat{S}$ of a separated uniform space S is compact if and only if the uniform structure is totally bounded [2, ch. II, p. 161]. Moreover, the completion is uniquely determined by the requirements that S be dense in $\hat{S}$ and that the embedding function ξ and its inverse ξ^{-1} (defined over $\xi(S) \subset \hat{S}$) both be uniformly continuous [2, ch. II, p. 154]. If we translate these statements to proximity language, we obtain the following result: *Every separated proximity space S admits a compactification in which S is dense, and which is uniquely determined by the requirement that the embedding function ξ and its inverse ξ^{-1} both be p-continuous or, what is equivalent,*

$$A \Subset B \Leftrightarrow \overline{\xi(A)} \subset \xi(B)^{\circ}\,. \tag{2.4}$$

This compactification is the *Smirnov compactification* of the given proximity space. [5, p. 557], [1, p. 181].

Conversely, we notice that every compactification of a (completely regular) topological space may be considered as a *completion*, namely as the completion determined by the totally bounded uniform structure induced from the uniform structure of the compact extension. Thus, if we translate to proximity structures: *Every compactification of a completely regular space is the Smirnov compactification of the space provided with a certain proximity structure, namely the one defined by* (2.4). [5, p. 557], [1, p. 181].

Finally, let $\mathscr{U}$ and $\mathscr{U}'$ be totally bounded uniform structures defined over S and S', respectively. By well-known theorems, a mapping of S into S' can be extended to a continuous mapping of the (compact) completion $\hat{S}$ into the completion $\hat{S}'$ if and only if it is uniformly continuous [2, ch. II, p. 151], [2, ch. II, p. 160].

This gives another of Smirnov's results: *A mapping of a proximity space into another can be extended to a continuous mapping of the Smirnov compactification of the former, into the Smirnov compactification of the latter if and only if it is p-continuous* [5, p. 557].

It appears that, in principle, an independent development of the theory of proximity structures is superfluous. Nevertheless, a direct reference to the axioms (P 1)–(P 6) may be convenient for certain applications involving uniform neighbourhoods. In particular we shall indicate some simple consequences of (P 1)–(P 6), which will throw some new light upon the interrelationship between proximity structures and totally bounded uniform structures.

We shall define a *base* of a proximity structure to be a system $\mathscr{B}$ of pairs (E, F) of subsets, such that $E \Subset F$, and such that

$$A \Subset B \Rightarrow \big(\exists (E, F) \in \mathscr{B}\big)[A \subset E,\ F \subset B]\,. \tag{2.5}$$

(Our definition differs from that given by A. Császár and S. Mrówka [3]. However, we believe ours to be the more adequate, since a base (2.5) uniquely determines the structure).

Let $\{S_\gamma, \mathscr{P}_\gamma, f_\gamma\}_{\gamma \in \Gamma}$ be a family of triplets, where S_γ is a set, $\mathscr{P}_\gamma$ is a proximity structure on S_γ, and f_γ is a mapping of some fixed set S into S_γ. By direct verification one may prove that the family of pairs

$$\Big(\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} f_{\gamma ij}^{-1}(A_{ij}),\quad \bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} f_{\gamma ij}^{-1}(B_{ij})\Big)\,, \tag{2.6}$$

$A_{ij} \Subset B_{ij}$ w.r. to $P_{\gamma_{ij}}$, will form a base for the *coarsest* proximity structure on S rendering all f_γ p-continuous. (According to Smirnov [5, p. 557], the structure $\mathscr{P}$ is *coarser* than $\mathscr{P}'$ if $A \Subset B$ (w.r. to $\mathscr{P}$) $\Rightarrow A \Subset B$ (w.r. to $\mathscr{P}'$)). Moreover, a mapping f of some other proximity space into S, is p-continuous (w.r. to this structure) if and only if all $f_\gamma \circ f$ are p-continuous (w.r. to $\mathscr{P}_\gamma$).

If in particular, $S = \Pi_{\gamma \in \Gamma} S_\gamma$ and $f_\gamma = pr_\gamma$, then (2.6) may be written:

$$\Big(\bigcup_{i=1}^{n} \prod_{\gamma \in \Gamma} pr_\gamma^{-1}(A_{\gamma,i}),\quad \bigcup_{i=1}^{n} \prod_{\gamma \in \Gamma} pr_\gamma^{-1}(B_{\gamma,i})\Big)\,, \tag{2.7}$$

$A_{\gamma,i} \Subset B_{\gamma,i}$ w.r. to $\mathscr{P}_\gamma$, where $A_{\gamma,i} = B_{\gamma,i} = S_\gamma$ for all but a finite number of indices. This particular proximity structure on $\Pi_{\gamma \in \Gamma} S_\gamma$ may be called the (initial) *product* of the original structures, and will be denoted $\Pi_{\gamma \in \Gamma} \mathscr{P}_\gamma$. It is worth noting that for a family of uniform spaces, the initial product of their derived proximity structures (1.1) is *coarser* than the proximity structure derived from the product of the uniform structures; if the original uniform structures are totally bounded, the two proximity structures on $\Pi_{\gamma \in \Gamma} S_\gamma$ are identical. However, on the Euclidean plane, $R \times R$, the initial product of the standard (metric) proximity structures on R and R is strictly coarser than the standard (metric) proximity structure on $R \times R$.

If we compare (1.2) and (2.7), we observe that *the entourages of the totally bounded uniform structure* $\mathscr{U}_\omega$ *corresponding to a proximity structure* $\mathscr{P}$, *are nothing else but the p-neighbourhoods of the diagonal* Δ *with respect to the product-structure* $\mathscr{P} \times \mathscr{P}$.

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