SUMMABILITY METHODS
AND UNBOUNDED SEQUENCES

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We wish to investigate the summability properties of regular matrices for unbounded sequences. The properties for bounded sequences have been described by Brudno [2] (see also [4] and [5]). The problem for unbounded sequences turns out to be markedly different. If a matrix $B = (b_{mn})$ sums a bounded sequence $\{s'_n\}$ that is not summable by $A = (a_{mn})$, then $B$ sums a non-enumerable set of independent bounded sequences that are not $A$ summable. We shall see that if $B$ sums an unbounded sequence $\{s'_n\}$ that is not $A$ summable, the sequences that are $B$ summable may all be of the form $\{Cs'_n + \sigma_n\}$ where $C$ is a constant and $\{\sigma_n\}$ is $A$ summable.

A matrix $A = (a_{mn})$; $m, n = 1, 2, \ldots$ is regular if the following conditions are fulfilled:

1° $\sum_n |a_{mn}| \leq H$ for every $m$;
2° $\lim_{m \rightarrow \infty} a_{mn} = 0$ for every $n$;
3° $\alpha_m = \sum_n a_{mn} \rightarrow 1$ as $m \rightarrow \infty$.

We shall consider regular matrices with finite rows, i.e. satisfying the following additional condition

4° \[ \begin{cases} a_{mn} = 0 & \text{when } n > \lambda(m), \\ a_{m, \lambda(m)} \neq 0. \end{cases} \]

Lemmas 1. If $A$ is a regular matrix satisfying condition 4° and

5° \[ \begin{cases} \lambda(m) = m, \\ a_{mn} = \begin{cases} 0 & \text{for } n < m - 1, \\ |a_{m, m-1} a_{mm}^{-1}| \geq K > 1 & \text{for } m \geq 2, \end{cases} \end{cases} \]

then every $A$ summable sequence has the form

$\{s_m\} = \{Cs'_m + \sigma_m\}$.

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where \( C \) is a constant, \( \{s'_m\} \) a certain unbounded sequence and \( \{\sigma_n\} \) a convergent sequence.

**Proof.** It will be convenient to use the notations

\[
\begin{align*}
  a_m &= a_{mm}, & A_m &= a_1a_2 \ldots a_m, & m = 1, 2, \ldots \\
  b_m &= a_{m,m-1}, & B_m &= b_2b_3 \ldots b_m, & m = 2, 3, \ldots
\end{align*}
\]

Note that conditions 3° and 5° imply that \( b_m \) is bounded away from 0; actually \( b_m > \frac{1}{2} \) from a certain \( m \).

That the sequence \( s_1, s_2, \ldots \) is \( A \) summable means that the sequence \( t_1 = a_1 s_1, \ t_2 = b_2 s_1 + b_3 s_2, \ t_3 = b_3 s_2 + b_3 s_3, \ldots \) converges to a limit \( s \). If we replace \( s_m \) by \( s_m - s \), \( t_m \) will be replaced by \( t_m - \alpha_m s \); hence, we may assume that \( t_m \to 0 \). From (1) follows

\[
\begin{align*}
  s_1 &= A_1^{-1} t_1, \\
  s_2 &= -A_2^{-1} B_2 (t_1 - A_1 B_2^{-1} t_2), \\
  s_3 &= A_3^{-1} B_3 (t_1 - A_1 B_2^{-1} t_2 + A_2 B_3^{-1} t_3),
\end{align*}
\]

and generally

\[
s_m = (-1)^{m-1} A_m^{-1} B_m (t_1 - A_1 B_2^{-1} t_2 + \ldots + (-1)^{m-1} A_{m-1} B_m^{-1} t_m) = (-1)^{m-1} A_m^{-1} B_m (t_1 - A_1 B_2^{-1} t_2 + A_2 B_3^{-1} t_3 - \ldots) -
\]

\[
(\frac{b_{m+1}^{-1}}{m+1} t_{m+1} - (a_{m+1} b_{m+1}^{-1}) b_{m+2}^{-1} t_{m+2} + (a_{m+1} b_{m+1}^{-1}) (a_{m+2} b_{m+2}^{-1}) b_{m+3}^{-1} t_{m+3} - \ldots).
\]

For the absolute value of the last term we have the upper bound

\[
(1 + K^{-1} + K^{-2} + \ldots) \max_{\mu > m} |b_\mu^{-1} t_\mu|,
\]

and since \( b_\mu^{-1} \) is bounded, this tends to zero. We have thus finished the proof of Lemma 1 with

\[
s'_m = (-1)^{m-1} A_m^{-1} B_m,
\]

\[
C = t_1 - B_2^{-1} t_2 + A_2 B_3^{-1} t_3 - A_3 B_4^{-1} t_4 + \ldots.
\]

We remark that every sequence \( \{s_m\} \) which satisfies the condition

\[
|s_{m-1}^{-1} s_m| \geq K > 1
\]

is \( A \) summable for some matrix \( A \) satisfying the conditions of Lemma 1. In fact, if we choose

\[
a_{11} = 1; \quad a_{mm} = -s_{m-1} (s_m - s_{m-1})^{-1}, \quad a_{m,m-1} = s_m (s_m - s_{m-1})^{-1},
\]

the conditions of Lemma 1 are satisfied, and we get \( t_m = 0, \ m = 2, 3, \ldots \).
DEFINITION. Let $A$ be a matrix. A matrix $B$ is called **stronger than $A$** if it sums all $A$ summable sequences, and it is called **strictly stronger than $A$** if it is stronger than $A$ and sums a sequence which is not $A$ summable.

**Theorem 1.** To a regular matrix $A$ satisfying $A^\circ$ corresponds a regular matrix $B$ satisfying $A^\circ$, strictly stronger than $A$, so that every matrix which is stronger than $A$ and sums a sequence which is $B$ summable but not $A$ summable, is stronger than $B$.

**Proof.** The matrix $A$ transforms a sequence $\{s_n\}$ into $\{t_m\}$ where

$$t_m = a_{m_1}s_1 + \ldots + a_{m_r}s_{\lambda(m)}.$$

The effect of a permutation of the rows of $A$ will be that the terms of $\{t_m\}$ are permuted in the same manner and this will not change the convergence properties of $\{t_m\}$. Therefore, we can assume that $\lambda(m)$ is increasing, i.e. that

$$\lambda(1) = \ldots = \lambda(m_1) < \lambda(m_1 + 1) = \ldots = \lambda(m_2) < \lambda(m_2 + 1) = \ldots.$$

For convenience, we put $m_0 = 0$. Correspondingly, we have a division of every transformed sequence $\{t_m\}$ in sections so that the terms

$$t_{m_{v-1} + 1}, \ldots, t_{m_v}$$

constitute the $v$th section.

We shall now construct a sequence $s'_1, s'_2, \ldots$ so that the transformed sequence $t'_1, t'_2, \ldots$ has the following property:

(2) $$|t'_m t'_n^{-1}| \geq K > 1$$

when

$$m_{v+1} < n \leq m_v, \quad m_v < m \leq m_{v+1}, \quad v \geq 1.$$

In order to do this, we choose $\{s'_n\}$ so that all terms are 0 except the terms $s'_{\lambda(m)}$. We first choose $s'_{\lambda(m_1)} \neq 0$. Next, we choose $s'_{\lambda(m_2)}$ so that the terms of the second section of $\{t'_m\}$ satisfy (2), and the construction proceeds by induction.

The next step of the proof is the construction of a matrix $D$ with the property that the set of $D$ summable sequences is identical with the set of all sequences $\{Ct'_m + u_m\}$ where $C$ is a constant whereas $\{t'_m\}$ is the sequence introduced above and $u_m$ is a convergent sequence.

The rows of the matrix $D$ will be indexed by pairs $(p,q)$ of numbers so that $p$ and $q$ correspond to adjacent sections, i.e.

$$m_{v-1} < p \leq m_v, \quad m_v < q \leq m_{v+1}, \quad v \geq 1.$$

Thus, the rows of $D$ fall in sections so that the $v$th section contains
\((m_{r} - m_{r-1})(m_{r+1} - m_{r})\) rows. The arrangement of the rows within the sections being of no importance for the summability properties, we may assume that the rows are arranged lexicographically with respect to \(p\) and \(q\).

Let \((d_{mn}) = (d_{(p,q),n})\) be chosen as follows:

\[
d_{(p,q),n} = \begin{cases} 
  t'_{q} (t'_{q} - t'_{p})^{-1} & \text{for } n = p, \\
  -t'_{p} (t'_{q} - t'_{p})^{-1} & \text{for } n = q, \\
  0 & \text{for } n \neq p,q.
\end{cases}
\]

It follows from (2) that \(D\) satisfies 1°, and it is obvious that \(D\) satisfies the conditions 2°, 3° and 4°. In particular \(D\) is regular so that every convergent sequence is \(D\) summable. It is clear that also \(\{t'_{m}\}\) is \(D\) summable, and hence that every sequence \(\{Ct'_{m} + u_{m}\}\), where \(C\) is constant and \(\{u_{m}\}\) convergent, is \(D\) summable.

Let \(\{t_{m}\}\) denote an arbitrary \(D\) summable sequence. We shall prove that \(\{t_{m}\}\) has the form \(\{Ct'_{m} + u_{m}\}\). We consider all possible sequences \(p_{1} < p_{2} < \ldots\) of integers so that \(\{t'_{p_{u}}\}\) contains exactly one term from each section of \(\{t'_{m}\}\). Let \(D_{p_{1}p_{2}} \ldots\) denote the matrix consisting of the rows of \(D\) with indices \(p_{1}, p_{2}, (p_{2}, p_{3}), \ldots\). The sequences \(\{t'_{p_{u}}\}\) and \(\{t_{p_{u}}\}\) are \(D_{p_{1}p_{2}} \ldots\) summable. Since this matrix satisfies the conditions of Lemma 1, all \(D_{p_{1}p_{2}} \ldots\) summable sequences have the form \(\{Ct''_{p_{u}} + u''_{p_{u}}\}\) where \(C\) is constant and \(\{u''_{p_{u}}\}\) converges. It follows in particular that

\[
t'_{p_{u}} = C*t''_{p_{u}} + u''_{p_{u}},
\]

where \(C* \neq 0\), as \(\{t'_{p_{u}}\}\) is unbounded. Hence

\[
t''_{p_{u}} = C*^{-1}t'_{p_{u}} - C*^{-1}u''_{p_{u}},
\]

and we have proved that all \(D_{p_{1}p_{2}} \ldots\) summable sequences have the form

\[
\{CC*^{-1}t'_{p_{u}} + (u''_{p_{u}} - C*^{-1}u''_{p_{u}})\}.
\]

We have thus proved that each of the subsequences \(\{t_{p_{u}}\}\) has the form \(\{Ct'_{p_{u}} + u''_{p_{u}}\}\). The constant \(C\) is uniquely determined by the condition that \(\{t'_{p_{u}} - Ct''_{p_{u}}\}\) is a bounded sequence. This implies that \(C\) is independent of the choice of \(p_{1}, p_{2}, \ldots\), since two subsequences with an infinity of common terms must correspond to the same value of \(C\). We can then write \(t_{m} = Ct'_{m} + u_{m}\) and \(\{u_{m}\}\) has the property that each of the subsequences \(\{u''_{p_{u}}\}\) converges, but this implies that \(\{u_{m}\}\) converges.

We can now prove Theorem 1 with \(B = DA\). Every sequence \(\{Cs'_{n} + v_{n}\}\) where \(\{v_{n}\}\) is \(A\) summable is by \(A\) transformed into \(\{Ct'_{m} + u_{m}\}\) where \(\{u_{m}\} = A\{v_{n}\}\) is convergent. The sequence \(\{Ct'_{m} + u_{m}\}\) is \(D\) summable, hence \(\{Cs'_{n} + v_{n}\}\) is \(B\) summable. It follows that \(B\) is strictly stronger than \(A\). On the other hand, let \(\{s_{n}\}\) be a \(B\) summable sequence. Then \(A\{s_{n}\}\) is
$D$ summable, hence $A\{s_n\} = \{Ct'_m + u_m\}$ where $u_m$ converges. We put $s_n = Cs'_n + v_n$, and it follows that 

$$A\{v_n\} = A\{s_n\} - CA\{s'_n\} = A\{s_n\} - C\{t'_m\} = \{u_m\},$$

hence $\{v_n\}$ is $A$ summable. Thus, if a matrix $B'$ sums every $A$ summable sequence and one sequence $\{Cs'_n + v_n\}$ with $C \neq 0$, then $B'$ sums $\{s'_n\}$ and, hence, every $B$ summable sequence.

**Definition.** Two sequences $\{s'_n\}$, $\{s''_n\}$ are called independent with respect to a matrix $A$ if no linear combination 

$$\{C's'_n + C''s''_n\} \text{ with } (C', C'') \neq (0, 0)$$

is $A$ summable.

Let $A$ be a given matrix. Theorem 1 states that there exists a matrix $B$ strictly stronger than $A$, so that a maximal system of $B$ summable sequences independent with respect to $A$ contains only one sequence. In this case the sequences which are $B$ summable but not $A$ summable are unbounded. In fact, if a matrix $B$ sums a bounded sequence which is not $A$ summable, there exists, according to Brudno ([2], see also [5]), a matrix $C$ strictly stronger than $A$ so that $B$ is strictly stronger than $C$. The following theorem is interesting in this connection:

**Theorem 2.** Let $A$ be a regular matrix satisfying $4^\circ$, and let $B$ denote a regular matrix stronger than $A$, so that there exist two $B$ summable sequences independent with respect to $A$. Then there exists a matrix $C$ strictly stronger than $A$ so that $B$ is strictly stronger than $C$.

**Proof.** According to the conditions of the theorem there exist two $B$ summable sequences $\{s'_n\}$ and $\{s''_n\}$ independent with respect to $A$, and we may even suppose that both sequences are $B$ summable with sum 0. The matrix $A$ transforms $\{s'_n\}$ into $\{t'_m\}$ and $\{s''_n\}$ into $\{t''_m\}$. No sequence $\{C't'_m + C''t''_m\}$, $(C', C'') \neq (0, 0)$, is convergent. Our proof will depend on the nature of the sequences $\{t'_m\}$, $\{t''_m\}$, but we shall start with some remarks which will be useful in all the particular cases.

We are going to choose certain subsequences $\{t'_{\mu_m}\}$ and $\{t''_{\mu_m}\}$ of $\{t'_m\}$ and $\{t''_m\}$. These subsequences are the transforms of $\{s'_n\}$ and $\{s''_n\}$ by the matrix $A^* = \{a_{\mu_mn}\}$ consisting of some of the rows of $A$.

Next, we choose a regular matrix $D$, which transforms one of the sequences $\{t'_{\mu_m}\}$, $\{t''_{\mu_m}\}$ into a sequence which does not converge to zero, while $D$ transforms a certain linear combination $\{C't'_{\mu_m} + C''t''_{\mu_m}\}$, $(C', C'') \neq (0, 0)$, into a sequence converging to zero. Then $\{C's'_n + C''s''_n\}$ is $DA^*$ summable with sum zero while $\{s'_n\}$ or $\{s''_n\}$ lacks this property. Since the matrix $D$ is regular every $A$ summable sequence is $DA^*$ sum-
mable. Finally we form a matrix $C$, which consists of all rows of $B$ and all rows of $DA^*$. All summable sequences and the sequence $\{C's'_n + C''s''_n\}$ are $C$ summable, hence $C$ is strictly stronger than $A$. On the other hand, every $C$ summable sequence is $B$ summable, and one of the sequences $\{s'_n\}$, $\{s''_n\}$ is $B$ summable, but not $C$ summable, hence $B$ is strictly stronger than $C$.

The proof of Theorem 2 will be finished when we have chosen $A^*$ and $D$ with the properties stated in the preceding section. We shall first assume that one of the sequences $\{t'_m\}$, $\{t''_m\}$, say $\{t'_m\}$ contains a subsequence $\{t'_{\mu_m}\}$ convergent to a limit $\neq 0$. We can choose this subsequence so that $t''_{\mu_m}$ tends to a finite limit or to infinity. We shall consider the two cases separately.

(i) If $t'_{\mu_m} \to u \neq 0$ and $t''_{\mu_m} \to v$, the sequence $\{us''_n - vs'_n\}$ is $A^*$ summable with sum 0, while $s'_n$ is $A^*$ summable with sum $\neq 0$. We may then choose $D$ as the unit matrix and the conditions will be satisfied.

(ii) If $t'_{\mu_m} \to u \neq 0$ and $|t''_{\mu_m}| \to \infty$, we may assume that

$$|t''_{\mu_{m+1}}| > 2|t''_{\mu_m}|.$$ 

According to the remark following Lemma 1, we can choose $D$ so that $\{t''_{\mu_m}\}$ is $D$ summable with sum 0 while $\{t'_m\}$ is $D$ summable with sum $u \neq 0$. With this choice the conditions will be satisfied.

In the remaining case we know that every convergent subsequence of $\{t'_m\}$ and $\{t''_m\}$ has limit 0. Assume first that we can choose $\{\mu_m\}$ so that one subsequence, say $\{t'_{\mu_m}\}$ converges to 0, while $\{t''_{\mu_m}\}$ diverges. We may then take $D$ as the unit matrix and the conditions will be satisfied. We shall now assume that it is impossible to choose $\{\mu_m\}$ in this way. We know that no sequence $\{C't'_m + C''t''_m\}$ converges. This property will be preserved if we delete all terms with $|t'_m| < 1$, $|t''_m| < 1$. When these terms are deleted, the absolute values of the terms of both sequences will tend to $\infty$. We can then choose $\{\mu_m\}$ so that $\{t'_{\mu_m}\}$ satisfies the condition

$$|t'_{\mu_{m+1}}| > 2|t'_{\mu_m}|.$$ 

If $\{C't'_{\mu_m} + t''_{\mu_m}\}$ does not converge for any $C'$, we can construct $D$ so that $\{t'_{\mu_m}\}$ is $D$ summable to 0 while $\{t''_{\mu_m}\}$ is not $D$ summable. If $\{C't'_{\mu_m} + t''_{\mu_m}\}$ converges to $t$ for some $C'$, then we consider the sequence $\{C't'_m + t''_m\}$. By the independence of $\{s'_m\}$ and $\{s''_m\}$ with respect to $A$, there exists an $\varepsilon > 0$ and a sequence $\{\nu_m\}$ such that

$$|C't''_m + t''_{\nu_m} - t| > \varepsilon, \quad \text{for all } m.$$ 

We then choose a subsequence $\{t'_{\nu_m}\}$ of $\{t_m\}$ satisfying (3) and containing
an infinite number of terms from each of the sequences \( \{t'_{\nu m}\} \) and \( \{t'_{\nu m}\} \). Then, finally, we can construct a matrix \( D \) which sums \( \{t'_{\nu m}\} \) to 0 but does not sum \( \{t''_{\nu m}\} \).

This completes the proof of the theorem.

We remark that any sequence \( \{s_{n}\} \), for which \( s_{n} = O(n^{\delta}) \) and is zero everywhere save for a subsequence \( \{n_{k}\} \) so that the counting function of \( \{n_{k}\} \) is \( o(n^{\delta}) \), is \((C, 1)\) summable (see Lorentz [3]). It follows that the iteration of \((C, 1)\) with itself sums a non-enumerable set of unbounded sequences that are not \((C, 1)\) summable though the set of bounded sequences is the same. In this case there exists a non-enumerable set of matrices of the type described in Theorem 2.

REFERENCES

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