MATH. SCAND. 7 (1959), 157–169

ON EXTRAPOLATING A POSITIVE DEFINITE FUNCTION FROM A FINITE INTERVAL

EDWIN J. AKUTOWICZ

1. Introduction. The datum of the problem studied in this paper is a complex-valued, continuous function $F(t)$ on the finite interval $(-A, A)$ which is positive definite on this interval. That is,

$$0 \leqq t_j,\ t_k < A \qquad \text{implies} \qquad \sum_{j,k} F(t_j - t_k)\, z_j \bar{z}_k \geqq 0$$

for arbitrary complex z's. Under these conditions M. Krein [1] showed in 1940 that $F(t)$ can be extended to a positive definite function $F_1(t)$ on $-\infty < t < \infty$. That is, according to Bochner's theorem, there exists an increasing bounded function $\sigma(\lambda)$, $\sigma(-\infty) = 0$, and normalized at its discontinuities, such that

$$F(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\sigma(\lambda)\,. \tag{1.1}$$

The extension $F_1(t)$ will not be unique in general [7], and it is the purpose of the present considerations to determine under what conditions there is a unique extrapolation, and to classify all solutions in the indeterminate case. The case of a unique extrapolation shall be referred to as the *determinate* case, following the usual terminology of the moment problem.

There are at least two fairly distinct categories of results on this problem. In the first category are those facts that can be obtained by using primarily aspects of the theory of Hilbert space. All of the results of the present paper are included here. The important fact that is taken advantage of is the existence of a certain non-negative quadratic form in infinitely many variables. This allows the construction of a Hilbert space H which is naturally related to our problem. The vectors of H are certain (equivalence classes of) functions of x, $-\infty < x < \infty$. An analogous quadratic form is also given in the classical power moment problem, where

---

Received December 21, 1958.

Presented at the International Congress of Mathematicians, Edinburgh, August, 1958. Also given as a lecture at the meeting on Fourier analysis at the University of Gothenburgh, November 29–30, 1958.

the relevant Hilbert space is built up starting with polynomials. In the present study, instead of polynomials, functions which are restrictions to the real axis of certain entire functions are used to construct H. A necessary and sufficient condition that the extrapolation be determinate is that the operator of "multiplication by x" be self-adjoint in H (that is, hypermaximal in H). The spectral analysis of a symmetric operator with deficiency index (1, 1) leads to the "limit circle" classification of the totality of extrapolations. This classification is of the same nature as that obtained by R. Nevanlinna [2] [3] for the power moment problem.

The second category refers to *sufficient* conditions for unique extrapolation. The methods here are much more function-theoretic. I hope to take up this subject on another occasion.

2. The Hilbert Space *H*. Consider the class H_0 of all functions $\varphi(u)$ that are of the form

$$\varphi(u) = \int_{-a}^{a} e^{iu\lambda}\, d\mu(\lambda), \qquad -\infty < u < \infty, \qquad a = \tfrac{1}{2}A\,,$$

where μ is an arbitrary complex-valued function of bounded variation. Put, for such functions φ_1 and φ_2,

$$(\varphi_1, \varphi_2) = \int_{-\infty}^{\infty} \varphi_1(u)\, \overline{\varphi_2(u)}\, d\sigma(u)\,, \tag{2.1}$$

where $\sigma(u)$ defines an arbitrary extrapolation through (1.1). Then (φ_1, φ_2) depends only upon the datum of the problem, namely upon the function $F(t)$, $-A<t<A$, and *not* upon any particular σ. For

$$\begin{aligned}(\varphi_1, \varphi_2) &= \int_{-\infty}^{\infty} d\sigma(u) \int_{-a}^{a} e^{iu\lambda_1}\, d\mu_1(\lambda_1) \int_{-a}^{a} e^{-iu\lambda_2}\, d\overline{\mu_2(\lambda_2)} \\ &= \int_{-a}^{a} d\mu_1(\lambda_1) \int_{-a}^{a} d\overline{\mu_2(\lambda_2)} \int_{-\infty}^{\infty} e^{i(\lambda_1-\lambda_2)u}\, d\sigma(u) \\ &= \int_{-a}^{a} d\mu_1(\lambda_1) \int_{-a}^{a} d\overline{\mu_2(\lambda_2)}\; F(\lambda_1-\lambda_2)\,.\end{aligned}$$

H_0 is a linear space in the obvious sense and we reduce H_0 modulo the subspace of functions φ_0 such that $(\varphi_0, \varphi_0)=0$. That is, vectors are taken not as individual functions φ but as classes of functions that coincide except on a set of σ-measure zero. The inner product (2.1) in H_0 can be transferred to the reduced space and the latter can be then completed to a Hilbert space H. In H vectors corresponding to functions in H_0 constitute a dense subset, and we shall regard H_0 itself as a dense subset of H. It is seen that H is obtained in an unambiguous way from the given positive definite function $F(t)$, $-A<t<A$.

3. The Operator T. In H_0 consider the operator T_0 given by

$$T_0\varphi_0(u) = u\varphi_0(u)$$

for those $\varphi_0 \in H_0$ such that

$$T_0\varphi_0 \in H_0 .$$

Such φ_0 are dense in H_0 and therefore in H. T_0 is obviously Hermitian ($T_0 \subset T_0^* =$ adjoint of T_0). Hence T_0^* is defined on a dense domain in H and hence there exists a minimal closed extension T of T_0, and T is also Hermitian[1].

LEMMA 3.1. *The deficiency index of T is either* (0, 0) *or* (1, 1).

PROOF. If the function that is identically 1 belongs to the range of $T-\lambda I$, $\operatorname{Im}\lambda \neq 0$, that is, if there exists φ_n belonging to $\mathscr{D}(T)\cap H_0$ such that

$$\|1 - (u-\lambda)\varphi_n(u)\| \to 0, \quad n \to \infty ,$$

then also

$$\|1 - (u-\bar\lambda)\overline{\varphi_n(u)}\| \to 0, \quad n \to \infty .$$

But since

$$\overline{\varphi_n(u)} = \int_{-a}^{a} e^{-ius}\, d\overline{\mu(s)} = \int_{-a}^{a} e^{ius}\, d[-\overline{\mu(-s)}] ,$$

we see that $\overline{\varphi_n(u)}$ also belongs to $\mathscr{D}(T)\cap H_0$. Hence 1 belongs also to the range of $T-\bar\lambda I$.

Suppose now that $\varphi(u)$ is bounded for $-\infty<u<\infty$ and is the restriction to the real axis of an entire function of exponential type $\leqq a$. Then for any fixed non-real z,

$$\frac{\varphi(u)-\varphi(z)}{u-z} \in L^2(-\infty, \infty) .$$

Hence, according to a well-known theorem of Paley and Wiener,

---

[1] Included in the specifications for a Hermitian operator T is the condition that its domain $\mathscr{D}(T)$ be dense in the relevant Hilbert space. $\mathscr{D}(T)$ and $\mathscr{R}(T)$ denote the domain and range of any operator T.

$$\frac{\varphi(u)-\varphi(z)}{u-z} = \int_{-a}^{a} e^{ius} X(s)\, ds, \qquad X \in L^2(-a, a).$$

That is,

$$\varphi(u) = \varphi(z) + (u-z)\int_{-a}^{a} e^{ius} X(s)\, ds.$$

This implies that the closed linear span of the range of $T-zI$ and the constant function 1 coincides with H. This fact, together with the preceding paragraph, proves Lemma 3.1.

4. Self-adjoint extensions of T determine extrapolations. One of the main results in the present theory is that every self-adjoint extension of T, including those obtained by enlarging the space H, determines a solution through its spectrum. To establish this it is convenient to make use of a theorem due essentially to M. S. Lifschitz [4] concerning the spectral decomposition of certain Hermitian operators. This "spectral theorem" employs a generalized (and in general non-unique) resolution of the identity, or spectral function, that is defined as follows.

DEFINITION. *A spectral function belonging to a closed Hermitian operator S in any Hilbert space $\boldsymbol{H}$ is any one-parameter family of bounded self-adjoint operators in $\boldsymbol{H}$,*

$$E(t), \qquad -\infty < t < \infty,$$

such that for every $\varphi \in \boldsymbol{H}$:

1. $\left(E(t)\varphi, \varphi\right)_{\boldsymbol{H}}$ *is non-decreasing for increasing t,*
2. $E(t-0)\varphi = E(t)\varphi$,
3. $E(t)\varphi \to 0$ *as $t \to -\infty$ and $E(t)\varphi \to \varphi$ as $t \to +\infty$,*
4. $E(t)$ *belongs to S in the sense that for $\varphi \in \mathscr{D}(S)$, $\psi \in \boldsymbol{H}$,*

$$(S\varphi, \psi)_{\boldsymbol{H}} = \int_{-\infty}^{\infty} t\, d\left(E(t)\varphi, \psi\right)_{\boldsymbol{H}},$$

and

$$\|S\varphi\|_{\boldsymbol{H}}^2 = \int_{-\infty}^{\infty} t^2\, d\left(E(t)\varphi, \varphi\right)_{\boldsymbol{H}}.$$

THEOREM 4.1 (Lifschitz). *If φ_1, φ_2, ε are elements of $\boldsymbol{H}$ such that*

$$\varphi_k - \varphi_k(x)\varepsilon = (S-xI)\psi_{k,x}, \qquad -\infty < x < \infty, \tag{4.1}$$

where $\varphi_k(x)$ is a continuous complex-valued function on $-\infty < x < \infty$ and

$$\psi_{k,x} \in \mathscr{D}(S) \quad \textit{for} \quad -\infty < x < \infty, \qquad k = 1, 2\ ,$$

then

$$(\varphi_1, \varphi_2)_{\boldsymbol{H}} = \int_{-\infty}^{\infty} \varphi_1(t)\overline{\varphi_2(t)}\, d\big(E(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}}\ ,$$

where $E(t)$ is any spectral function belonging to S.

A sketch of the proof runs as follows. First, suppose S is self-adjoint in $\boldsymbol{H}$, in which case $E(t)$ is a unique orthogonal family of projections such that $E(t_1)E(t_2) = E\big(\min(t_1, t_2)\big)$. Hence it follows for any interval Δ that

$$E(\Delta)\, S\psi_x = E(\Delta)\int_{-\infty}^{\infty} t\, dE(t)\,\psi_x = \int_{\Delta} t\; dE(t)\,\psi_x\ .$$

By (4.1), dropping the subscript k,

$$\begin{aligned} E(\Delta)\varphi - \varphi(x)E(\Delta)\varepsilon &= \int_{\Delta} t\; dE(t)\,\psi_x - x\int_{\Delta} dE(t)\,\psi_x \\ &= \int_{\Delta} (t-x)\; dE(t)\,\psi_x\ . \end{aligned}$$

Hence if $x \in \Delta$,

$$\|E(\Delta)\varphi - \varphi(x)E(\Delta)\varepsilon\| \leqq \text{length}\ \Delta \cdot \|E(\Delta)\psi_x\|\ .$$

Hence if x_0 belongs to the point spectrum of S choose Δ as the half-open interval $x_0 \leqq x < x_0 + \delta$ and allow $\delta \to 0$ to conclude that

$$\big(E(x_0+0) - E(x_0)\big)\varphi = \varphi(x_0)\big(E(x_0+0) - E(x_0)\big)\varepsilon\ . \tag{4.2}$$

Hence if S has pure point spectrum the conclusion follows by addition from (4.2) and the orthogonal character of $E(t)$. Therefore assume that S has only continuous spectrum. Let Δ_0 be any fixed finite interval and $\Delta_0 = \Delta_1 \cup \ldots \cup \Delta_n$ a partition of Δ_0. Then, for $\xi_\nu \in \Delta_\nu$,

$$\begin{aligned} E(\Delta_0)\varphi &- \int_{\Delta_0} \varphi(t)\; dE(t)\varepsilon \\ &= \sum_{\nu=1}^{n} \Big(E(\Delta_\nu)\varphi - \int_{\Delta_\nu} \varphi(t)\; dE(t)\varepsilon\Big) \\ &= \sum_{\nu=1}^{n} \big(E(\Delta_\nu)\varphi - \varphi(\xi_\nu)E(\Delta)\varepsilon\big) + \sum_{\nu=1}^{n} \Big(\varphi(\xi_\nu)E(\Delta_\nu)\varepsilon - \int_{\Delta_\nu} \varphi(t)\; dE(t)\varepsilon\Big) \\ &= \sum_{\nu=1}^{n} \int_{\Delta_\nu} (t-\xi_\nu)\; dE(t)\,\psi_{\xi_\nu} + \sum_{\nu=1}^{n} \int_{\Delta_\nu} \big(\varphi(\xi_\nu) - \varphi(t)\big)\; dE(t)\varepsilon\ . \end{aligned}$$

As

$$\max_{\nu} \text{ length } \Delta_\nu \to 0$$

the last two sums tend to 0 strongly in $\boldsymbol{H}$. Hence

$$E(\Delta_0)\varphi = \int_{\Delta_0} \varphi(t)\, dE(t)\varepsilon$$

for any finite interval Δ_0. It follows by passing to the limit $\Delta_0 \to (-\infty, \infty)$ that

$$\varphi = \int_{-\infty}^{\infty} \varphi(t)\, dE(t)\varepsilon\,.$$

This implies (by the orthogonal character of $E(t)$) that

$$(\varphi_1 \cdot \varphi_2)_{\boldsymbol{H}} = \int_{-\infty}^{\infty} \varphi_1(t)\, \overline{\varphi_2(t)}\, d\big(E(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}}\,.$$

Thus the theorem is proved if S is self-adjoint.

Second, according to a theorem of Neumark [6], for every spectral function $E(t)$ of S there exists a self-adjoint operator S' in a larger Hilbert space $\boldsymbol{H}' \supset \boldsymbol{H}$ such that

$$\varphi \in \mathscr{D}(S) \qquad \text{implies} \qquad S'\varphi = S\varphi\,,$$

and

$$E(t) = P_{\boldsymbol{H}} E'(t), \qquad -\infty < t < \infty\,,$$

where $P_{\boldsymbol{H}}$ is the projection in $\boldsymbol{H}'$ on $\boldsymbol{H}$ and $E'(t)$ is the unique resolution of the identity belonging to S' in $\boldsymbol{H}'$. By what has already been proved,

$$(\varphi_1, \varphi_2)_{\boldsymbol{H}'} = \int_{-\infty}^{\infty} \varphi_1(t)\, \overline{\varphi_2(t)}\, d\big(E'(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}'}\,, \tag{4.3}$$

where the orthogonality of $E'(t)$ is used. But by assumption φ_1, φ_2, ε belong to $\boldsymbol{H}$. Therefore

$$\big(E'(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}'} = \big(E'(t)\varepsilon, P_{\boldsymbol{H}}\varepsilon\big)_{\boldsymbol{H}'} = \big(P_{\boldsymbol{H}}E'(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}'} = \big(E(t)\varepsilon, \varepsilon\big)_{\boldsymbol{H}}\,,$$

and this, by (4.3), completes the proof.

Now consider the Hermitian operator T acting in H.

THEOREM 4.2. *If $E(t)$ is any spectral function belonging to T then an extrapolation is generated through* (1.1) *by* $\sigma(t) = \big(E(t)1, 1\big)_{\boldsymbol{H}}$.

PROOF. Observe that functions $\varphi(u)$ of the form

$$\varphi(u) = \int_{-a}^{a} e^{ius} X(s)\, ds\,,$$

where X is of bounded total variation over $(-a, a)$ and $X(a) = X(-a) = 0$, belong to $\mathscr{D}(T)$. For, by an integration by parts,

$$u\varphi(u) = -i \int_{-a}^{a} iu\, e^{ius} X(s)\, ds = \int_{-a}^{a} e^{ius}\, d[iX(s)],$$

so that $T\varphi$ belongs to H.

Now consider the identity

$$e^{itu} - e^{itx} = (u-x)\, \frac{e^{itu} - e^{itx}}{u-x}\,, \tag{4.4}$$

where x, t are real parameters, $-a < t < a$, $-\infty < x < \infty$. It is easy to check the formula

$$\int_{-\infty}^{\infty} \frac{e^{itu} - e^{itx}}{u-x}\, e^{-isu}\, du = X_{x,t}(s)\,,$$

where

$$X_{x,t}(s) = \begin{cases} 0 & \text{if } s \text{ lies not between 0 and } t\,, \\ e^{i(t-s)x} & \text{if } s \text{ lies between 0 and } t\,. \end{cases}$$

After a Fourier inversion, according to the preceding paragraph,

$$\frac{e^{itu} - e^{itx}}{u-x} \in \mathscr{D}(T)\,.$$

Put, for $-\infty < u < \infty$,

$$\begin{aligned} \varphi_1(u) &= e^{itu}, \qquad -a < t < a\,, \\ \varphi_2(u) &= 1\,, \\ \varepsilon(u) &= 1\,, \\ \psi_{1,x}(u) &= \frac{e^{itu} - e^{itx}}{u-x}\,, \\ \psi_{2,x}(u) &= 0\,. \end{aligned}$$

In view of the identity (4.4) the conditions of Lifschitz's theorem are satisfied. It follows that

$$(e^{itu}, 1)_H = \int_{-\infty}^{\infty} e^{itu}\, d(E(u)1, 1)_H\,,$$

where $E(u)$ is any spectral function of T. On the other hand,

$$(e^{itu}, 1)_H = F(t), \qquad -a < t < a\,.$$

Therefore $\sigma(u) = (E(u)1, 1)_H$ does indeed determine an extrapolation.

In section 6 we shall show that every solution is of this form.

5. A necessary and sufficient condition for determinacy. It is now easy to state the principal result about the determinate case.

THEOREM 5.1. *The extrapolation problem is determinate if and only if T is self-adjoint in H.*

PROOF. Let it first be assumed that the determinate case occurs and that T is not self-adjoint in H. Then the deficiency index of T must be $(1, 1)$ by Lemma 3.1, and there exist two different extensions of T to self-adjoint operators A_1 and A_2 in H. (Actually such self-adjoint extensions within H are in one-to-one correspondence with the points on the periphery of the unit circle).

Let it be recalled how such extensions are constructed (von Neumann). Suppose S denotes any closed Hermitian operator in H and V its Cayley transform,

$$V = (S - zI)(S - z^*I)^{-1}, \qquad \operatorname{Im} z > 0 ,$$

where the asterisk indicates the complex conjugate. The domain and the range of V are given by

$$\mathscr{D}(V) = \mathscr{R}(S - z^*I) \qquad \text{and} \qquad \mathscr{R}(V) = \mathscr{R}(S - zI) ,$$

respectively. Self-adjoint extensions of S exist in H if and only if the dimensions of the deficiency subspaces coincide:

$$\dim(H \ominus \mathscr{R}(S - z^*I) = \dim(H \ominus \mathscr{R}(S - zI) .$$

Assuming this, let

$$V' \colon \; H \ominus \mathscr{R}(S - z^*I) \to H \ominus \mathscr{R}(S - zI)$$

be an isometric correspondence. Then V' extends V to a unitary operator V_0 in H. The Cayley transform of V_0 is a self-adjoint operator S_0 in H,

$$S_0 = (z^* V_0 - zI)(V_0 - I)^{-1} ,$$

and S_0 is an extension of S. All self-adjoint extensions of S in H are obtainable in this way.

Returning to the operator T, it follows from the proof of Lemma 3.1 that the constant $1 \in H$ has non-zero projections 1_{z^*} and 1_z in $H \ominus \mathscr{R}(T - z^*I)$ and $H \ominus \mathscr{R}(T - zI)$, respectively, and these two projections span the two one-dimensional deficiency subspaces. Writing $\eta_{z^*} = 1_{z^*}/\|1_{z^*}\|$,

$$V_1 = (A_1 - zI)(A_1 - z^*I)^{-1}, \qquad V_2 = (A_2 - zI)(A_2 - z^*I)^{-1},$$

we must have

$$V_1\eta_{z^*} = \beta V_2 \eta_{z^*}, \ |\beta| = 1, \ \beta \neq 1 .$$

Also

$$H = \big(H \ominus \mathscr{R}(T - z^*I)\big) \oplus \mathscr{R}(T - z^*I)$$

implies

$$1 = (1, \eta_{z^*})\eta_{z^*} + \theta, \qquad \theta \in \mathscr{R}(T - z^*I) .$$

Since $V_1\theta = V_2\theta$, it follows that

$$\begin{aligned} V_1 1 - V_2 1 &= (1, \eta_{z^*})(V_1\eta_{z^*} - V_2\eta_{z^*}) \\ &= (1, \eta_{z^*})(\beta - 1)V_2\eta_{z^*} . \end{aligned}$$

Now the resolvents $R_1 = (A_1 - z^*I)^{-1}$ and $R_2 = (A_2 - z^*I)^{-1}$ satisfy

$$R_1 - R_2 = (z^* - z)^{-1}(V_1 - V_2) .$$

Hence

$$(R_1 1, 1) - (R_2 1, 1) = (z^* - z)^{-1}(1, \eta_{z^*})(\beta - 1)(V_2\eta_{z^*}, 1) \neq 0 .$$

Hence $\big(E_1(u)1, 1\big)$ and $\big(E_2(u)1, 1\big)$, where $E_1(u)$ and $E_2(u)$ are the spectral functions of A_1 and A_2, must be different functions of u. By theorem 4.2 these functions would then determine different extrapolations, which is a contradiction. Therefore if the determinate case occurs, T is necessarily self-adjoint in H.

Conversely, assume T is self-adjoint in H. Then there will exist $\varphi_n \in \mathscr{D}(T_0)$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |1 - (u - z)\varphi_n(u)|^2 \, d\sigma(u) = 0, \qquad \operatorname{Im} z \neq 0 .$$

But

$$\int_{-\infty}^{\infty} |(u - z)^{-1} - \varphi_n(u)|^2 \, d\sigma(u) \leqq \max_{-\infty < u < \infty} |u - z|^{-2} \int_{-\infty}^{\infty} |1 - (u - z)\varphi_n(u)|^2 \, d\sigma(u) .$$

Hence $(u - z)^{-1}$ belongs to H as a function of u. By the spectral form of the resolvent of T,

$$\big((u - z)^{-1}, 1\big)_H = \int_{-\infty}^{\infty} (u - z)^{-1} \, d(E(u)1, 1)_H ,$$

and by the definition of $(\ , \ )_H$, for any solution σ,

$$\big((u - z)^{-1}, 1\big)_H = \int_{-\infty}^{\infty} (u - z)^{-1} \, d\sigma(u).$$

Here $E(u)$ is the unique orthogonal spectral family of T. Hence by Stieltjes' inversion formula,

$$\sigma(u) = \big(E(u)1,\ 1\big)_H .$$

As $E(u)$ is constructed entirely within H, this means that σ is unique. Q. E. D.

6. The classification of the totality of extrapolations. In the indeterminate case the set of all solutions possesses a classification which is substantially identical with that obtained by R. Nevanlinna in 1922 in the indeterminate case of the power moment problem. Nevanlinna solves the latter problem by determining all holomorphic functions of a certain class possessing the same asymptotic expansion that is defined in terms of the given moments. The present classification is effected by using the known form of all self-adjoint extensions of Hermitian operators with deficiency index (1, 1). These self-adjoint extensions include those where the extended operator acts in a larger Hilbert space containing the given Hilbert space[2].

There are two steps to be taken in the classification problem:

(i) Every solution $\sigma(t)$ must be shown to arise from a spectral function belonging to T in the form $\sigma(t) = (E(t)1,\ 1)_H$.

(ii) The totality of spectral functions must be described.

ad (i). Suppose $\sigma(t)$ is any solution. Consider the particular Hilbert space H^+ $\big(=H^+(\sigma)\big)$ consisting of all (equivalence classes of) Borel functions $\varphi(t)$, $-\infty < t < \infty$, for which

$$\int_{-\infty}^{\infty} |\varphi(t)|^2\, d\sigma(t) < \infty ,$$

and for which

$$(\varphi_1, \varphi_2)_{H^+} = \int_{-\infty}^{\infty} \varphi_1(t)\, \overline{\varphi_2(t)}\, d\sigma(t) .$$

Then $H^+ \supset H$. For every t, $-\infty < t < \infty$, the quantity

$$B_t(\varphi_1, \varphi_2) = \int_{-\infty}^{t} \varphi_1(u)\, \overline{\varphi_2(u)}\, d\sigma(u)$$

defines a bilinear form in H^+ and therefore a projection $G(t)$ in H^+:

[2] A theory of extensions of Hermitian operators was developed during the 1940's by M. Krein and M. A. Neumark, and a useful exposition of part of their results can be found in the excellent book [6] of N. Achieser and M. A. Glasmann. We refer the reader particularly to Supplement I of this book.

$$B_t(\varphi_1, \varphi_2) = \big(G(t)\varphi_1, \varphi_2\big) .$$

Of course $G(t)$ depends upon σ. This family of projections $G(t)$, $-\infty < t < \infty$, is a resolution of the identity in H^+ in the ordinary sense belonging to the self-adjoint operator of multiplication by the independent variable. Putting

$$E(t) = P_H G(t), \qquad -\infty < t < \infty ,$$

where P_H is the projection in H^+ on H, we obtain a spectral function in H. This spectral function belongs to T, for if $\varphi \in \mathscr{D}(T)$ and $\psi \in H$, then

$$\begin{aligned}(T\varphi, \psi)_H &= \int_{-\infty}^{\infty} t\varphi(t)\overline{\psi(t)}\, d\sigma(t) \\ &= \int_{-\infty}^{\infty} t\, d \int_{-\infty}^{t} \varphi(u)\, \overline{\psi(u)}\, d\sigma(u) \\ &= \int_{-\infty}^{\infty} t\, d\big(G(t)\varphi, \psi\big)_{H^+} \\ &= \int_{-\infty}^{\infty} t\, d\big(G(t)\varphi, P_H\psi\big)_{H^+} \\ &= \int_{-\infty}^{\infty} t\, d\big(E(t)\varphi, \psi\big)_H ,\end{aligned}$$

and similarly

$$\|T\varphi\|_H{}^2 = \int_{-\infty}^{\infty} t^2\, d\big(E(t)\varphi, \varphi\big)_H .$$

Now it is easy to show that $\sigma(t) = (E(t)1, 1)_H$. For the resolvent R_z of the self-adjoint operator of multiplication by the independent variable in H^+ satisfies

$$(R_z 1, 1)_{H^+} = \int_{-\infty}^{\infty} \frac{1}{u-z}\, d\big(G(u)1, 1\big)_{H^+}$$

and also

$$(R_z 1, 1)_{H^+} = \int_{-\infty}^{\infty} \frac{1}{u-z}\, d\sigma(u)$$

for every non-real z. Hence $\sigma(t) = \big(G(t)1, 1\big)_{H^+} = \big(E(t)1, 1\big)_H$. This establishes (i).

ad (ii). There is a one-to-one correspondence between the class of all spectral functions belonging to the closed Hermitian operator T and the class of all generalized resolvents of the same operator. Such a correspondence is implemented by the relation

$$(R_z\varphi, \psi) = \int_{-\infty}^{\infty} (t-z)^{-1}\, d(E(t)\varphi, \psi), \qquad \operatorname{Im} z \neq 0, \quad \varphi, \psi \in H,$$

between a generalized resolvent R_z and a spectral function $E(t)$ belonging to T. Thus it suffices for our purposes to describe the totality of generalized resolvents of T. As T has deficiency index $(1, 1)$ this can be done explicitly as follows [6]. All inner products are in H.

Every generalized resolvent R_z of T is of the form

$$R_z = R'_z - (\cdot, \eta_z)(f(z)+q_0(z))^{-1}\eta_{z^*}, \tag{6.1}$$

where the vectors η_{z^*}, η_z have their previous meaning (section 5), $q_0(z)$ is a certain fixed function of z in the upper half-plane, and $f(z)$ is holomorphic in the upper half-plane with non-negative imaginary part there. R'_z is any fixed generalized resolvent of T. As f varies subject to the conditions stated, R_z varies over all generalized resolvents of T.

This classification can be given a familiar geometrical form by applying (6.1) to $1 \in H$ and forming the inner product with 1:

$$\begin{aligned}(R_z 1, 1) &= (R'_z 1, 1) - (1, \eta_z)(\eta_{z^*}, 1)(f(z)+q_0(z))^{-1} \\ &= (p_1(z)f(z)+p_0(z))(f(z)+q_0(z))^{-1}\end{aligned} \tag{6.2}$$

where p_0, p_1, q_0 are certain fixed functions of non-real z. Choosing $f(z)=t=$ real constant, identically in z, it follows by (6.2) that the complex number

$$(R_z 1, 1)$$

for fixed z varies over a circle $\Gamma(z)$ in the upper half-plane as t varies over all real values. These peripheral points in 1-1 correspondence with the totality of self-adjoint extensions of the first kind; that is, extensions within H as described in section 5. As the set of all extensions is clearly convex, these facts imply that as

$$(R_z 1, 1)$$

runs over the disk bounded by $\Gamma(z)$, R_z runs over all generalized resolvents of T.

REFERENCES

1. M. Kreĭn, *Sur le problème du prolongement des fonctions hermitiennes positives et continues*, Comptes Rendus (Doklady) Acad. Sci. URSS (N. S.) 26 (1940), 17–22.
2. R. Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem*, Ann. Acad. Sci. Fennicae (A) 18 (1922), no. 5.
3. R. Nevanlinna, *Über beschränkte analytische Funktionen*, Ann. Acad. Sci. Fennicae (A) 32 (1929), no. 7. (= Commentationes in honorem E. L. Lindelöf, no. 7).
4. M. S. Livshitz, *On an application of the theory of Hermitian operators to the generalized problem of moments*, Comptes Rendus (Doklady) Acad. Sci. URSS (N. S.) 44 (1944), 3–7.
5. M. Kreĭn and M. Krasnoselskiĭ, *Fundamental theorems on extension of Hermitian operators and certain of their applications to the theory of orthogonal polynomials and the problem of moments*, Uspehi Mat. Nauk 2 (1947), 60–106. (Russian.)
6. N. Achieser and M. A. Glasmann, *Theorie der Linearen Operatoren im Hilbert-Raum*, Berlin, 1954.
7. C.-G. Esseen, *Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law*, Acta Math. 77 (1945), 1–125.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASS., U. S. A.
AND
UNIVERSITY OF UPPSALA, SWEDEN