MATH. SCAND. 7 (1959), 5–32

LOCALLY COGENT BOUNDARY OPERATORS

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Let V be a domain with boundary S in the $(m+1)$-dimensional Euclidean space R^{m+1} with points $(t, x_1, \ldots, x_m)$ and L a linear differential operator with constant coefficients. Let $C_0^\infty(V)$ be the set of infinitely differentiable functions with compact supports in V and let $\|\cdot\|$ denote the usual L^2 norm in V. The following theorem is valid (Malgrange [6], Hörmander [4]).

THEOREM 0.1. *There exists a constant C such that for every $u \in C_0^\infty(V)$,*

$$\|u\| \leqq C\|Lu\| . \tag{0.1}$$

This theorem does not give any information for boundary problems, since it is only concerned with functions vanishing in a neighbourhood of the boundary S of V. It is evident that a corresponding result is not valid if all restrictions at the boundary are omitted and functions in the set $C^\infty(\overline{V})$ of infinitely differentiable functions in the closure $\overline{V}$ of V are considered, since we can always find solutions in $C^\infty(\overline{V})$ to the equation $Lu=0$ which are not $\equiv 0$ in V. Evidently no such function satisfies (0.1).

It is natural to inquire for which sets l of linear boundary operators there is an inequality of the type (0.1) for all $u \in C^\infty(\overline{V})$ with $lu=0$. In this case the boundary operator l will be called weakly cogent for the interior differential operator L.

Introducing an operator $\mathscr{L}=(L, l)$ and a suitable norm in the corresponding product space it is possible also to treat the inhomogeneous boundary problem and to investigate if the boundary operator has the property that there exists a constant C such that for all $u \in C^\infty(\overline{V})$

$$\|u\| \leqq C\|\mathscr{L}u\| .$$

The boundary operator l will then be called a cogent boundary operator for the interior differential operator L.

For a general differential operator L and a general region V it is not easy to decide which boundary operators l are cogent. Therefore, the

Received February 15, 1959.

problem is here restricted to homogeneous operators L and considered locally for a part S_0 of the boundary which for the sake of simplicity is supposed to be plane. All functions considered are supposed to belong to the set $C_0^\infty(V \cup S_0)$ of functions in $C^\infty(\bar{V})$ which vanish outside compact subsets of $V \cup S_0$. In this case the term locally cogent is used.

Our main result is concerned with a boundary operator of Dirichlet's type on S_0, which is supposed to be a subset of the plane $t=0$, that is

$$lu = \left(u, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{p-1} u}{\partial t^{p-1}}\right). \tag{0.2}$$

The characteristic polynomial $L(\xi, \tau)$ of L is considered as a polynomial in τ for fixed ξ. We show that the operator (0.2) is locally cogent for the differential operator L on S_0 if and only if the number of zeros of $L(\xi, \tau)$ with positive imaginary parts is not greater than p for any ξ.

In the case of two dimensions ($m=1$) and an operator L with simple characteristics, results are obtained also for boundary operators with constant coefficients which are more general than the Dirichlet operator (0.2).

For functions $u \in C_0^\infty(V)$, Hörmander [4] determined the operators M which can be estimated by means of L in the sense that there exists a constant C such that

$$\|Mu\| \leqq C\|Lu\|, \qquad u \in C_0^\infty(V).$$

These operators are called weaker than L. It might be expected that in the case of locally cogent boundary operators, it would be possible to estimate all operators M which are weaker than L by means of L when $u \in C_0^\infty(V \cup S_0)$ and $lu=0$. It is shown by examples that this is not so.

In the case of elliptic operators L, the problem of finding cogent boundary operators has been studied under the name of "the coerciveness problem" by Aronszajn [2], Schechter [8] [9], and Agmon [1]. Here it proves possible to estimate all operators M, weaker than L, which in this case are all operators of order $\leqq$ the order of L. A closely connected problem has been studied by Hörmander [5].

The plan of the paper is the following:

In Section 1 notations and definitions are introduced and the relation between locally cogent boundary operators and correctly posed boundary problems is established (Theorem 1.1). Further a lemma which is used in Sections 2 and 3 is proved.

In Section 2, boundary operators of Dirichlet's type are studied. The results are divided into a necessity part, Theorem 2.1, and a sufficiency part, Theorem 2.2. In the former, the homogeneous boundary problem is treated, whereas the estimates obtained in the latter are valid also in the inhomogeneous case. Examples are also given of operators M and $\mathscr{L}=(L, l)$ where l is locally cogent for L and M is weaker than L, but where Mu can not be estimated by means of Lu when $u \in C_0^\infty(V \cup S_0)$ and $lu=0$.

In Section 3, the two-dimensional case with more general homogeneous boundary operators is considered for an operator L with simple characteristics. Also here the results are divided into a necessity part, Theorem 3.1, and a sufficiency part, Theorem 3.2.

We find it convenient in our arguments to let C denote a positive constant. It need not, however, always denote the same constant even in the course of a particular proof. When necessary we distinguish between different constants by using subscripts.

I wish to thank Professor L. Hörmander for several instructive and stimulating discussions, and Professor W. Donoghue for revising the English manuscript.

1. Preliminaries. Let V be a region in the $(m+1)$-dimensional Euclidean space R^{m+1} with points $x=(x_0, x_1, \ldots, x_m)$ and L a differential operator with constant coefficients

$$L = L(D) = \sum_{|\alpha| \leq n} L_\alpha D^\alpha .$$

Here $\alpha=(\alpha_0, \ldots, \alpha_m)$, where α_i are non-negative integers,

$$|\alpha| = \sum_{i=0}^{m} \alpha_i, \quad \text{and} \quad D^\alpha = D_0^{\alpha_0} \ldots D_m^{\alpha_m} \quad \text{with} \quad D_k = -i \frac{\partial}{\partial x_k} .$$

Let the boundary of V be S, and denote by $C^\infty(\overline{V})$ the set of restrictions to the closure $\overline{V}=V \cup S$ of V of complex-valued, infinitely differentiable functions in R^{m+1}. The operator L can then be considered as an operator from $C^\infty(\overline{V})$ to $C^\infty(\overline{V})$.

Suppose that S consists of N smooth pieces S_i, $i=1, \ldots, N$, and that with each S_i are associated σ_i differential operators with constant coefficients,

$$l_{ik} = l_{ik}(D) = \sum_{\alpha} l_{ik,\alpha} D^\alpha, \qquad k = 1, \ldots, \sigma_i; \; i = 1, \ldots, N .$$

We denote by $C^\infty(S_i)$ the restriction to S_i of functions in $C^\infty(\overline{V})$. The operator l_{ik} can then be considered as an operator from $C^\infty(\overline{V})$ to $C^\infty(S_i)$.

We reduce the differential operators l_{ik}, $k=1, \ldots, \sigma_i$, into an operator $l_i=(l_{i1}, \ldots, l_{i,\sigma_i})$ from $C^\infty(\overline{V})$ to the set $C^\infty(S_i, \sigma_i)$ of vectors with σ_i components in $C^\infty(S_i)$, and the operators l_i, $i=1, \ldots, N$, into an operator $l=(l_1, \ldots, l_N)$ from $C^\infty(\overline{V})$ to the product space

$$C^\infty(S, \sigma) = C^\infty(S_1, \sigma_1) \times \ldots \times C^\infty(S_N, \sigma_N), \quad \sigma = (\sigma_1, \ldots, \sigma_N) .$$

Finally we reduce the operators L and l into an operator $\mathscr{L} = (L, l)$ from $C^\infty(\bar{V})$ to the product space $C^\infty(V, S, \sigma) = C^\infty(\bar{V}) \times C^\infty(S, \sigma)$. With these notations a boundary problem can be written as an equation

$$\mathscr{L}u = \mathscr{F} , \tag{1.1}$$

where $\mathscr{F}$ is some element in $C^\infty(V, S, \sigma)$. In the sequel we shall refer to L as the interior operator and to l and its components as the boundary operators.

$\mathscr{L}$ can now be regarded as a transformation between two normed linear spaces if we provide $C^\infty(\bar{V})$ with a norm $\|\cdot\|_1$ and $C^\infty(V, S, \sigma)$ with a norm $\|\cdot\|_2$. We shall be interested in such boundary problems (1.1) for which the operator $\mathscr{L}$ satisfies the condition in the following definition.

Definition 1.1. *The boundary operator l is cogent for the interior operator L with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, if there exists a constant C such that*

$$\|u\|_1 \leqq C \|\mathscr{L}u\|_2, \quad u \in C^\infty(\bar{V}) .$$

Since the problem of characterizing cogent boundary operators in general offers great difficulties, we restrict the problem and study it locally on a given part S_i of the boundary. We therefore consider the set $C_0^\infty(V \cup S_i)$ of functions in $C^\infty(\bar{V})$ which vanish outside compact subsets of $V \cup S_i$ and make the following definition.

Definition 1.2. *The boundary operator l_i is locally cogent on S_i for the interior operator L with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ if there exists a constant C such that*

$$\|u\|_1 \leqq C \|\mathscr{L}u\|_2, \quad u \in C_0^\infty(V \cup S_i) .$$

For the boundary operator $lu = (l_1 u, \ldots, l_N u)$ to be cogent for L, it is, of course, necessary for l_i to be locally cogent on S_i for L, $i = 1, \ldots, N$. But easily constructed examples show that this is by no means sufficient.

For our investigations we will make use of the following quadratic norms (dV is the element of volume in V and dS the element of surface area on S).

$$\|\varphi\|_V = \left\{ \int_V |\varphi|^2 dV \right\}^{\frac{1}{2}}, \quad \|\varphi\|_{S_i} = \left\{ \int_{S_i} |\varphi|^2 dS \right\}^{\frac{1}{2}},$$

and

$$\|\varphi\|_{R,p} = \left\{ \sum_{|\alpha| \leqq p} \|D^\alpha \varphi\|_R^2 \right\}^{\frac{1}{2}}, \quad R = V, S_i ,$$

where in case $R = S_i$ the derivation symbol D^α is understood to refer to derivations after directions in S_i only.

For $\|\cdot\|_1$ we shall use norms of the type

$$\|u\|_1 = \|u\|_{V,p} , \tag{1.2}$$

where p usually will be $=0$, and for $\|\cdot\|_2$ we shall consider norms such as

$$\|\mathscr{L}u\|_2 = \left\{ \|Lu\|_V{}^2 + \sum_{i=1}^{N} \sum_{k=1}^{\sigma_i} \|l_{ik}u\|_{S_i,p_{ik}}{}^2 \right\}^{\frac{1}{2}} . \tag{1.3}$$

In analogy with the Definitions 1.1 and 1.2 we introduce the following ones for homogeneous boundary problems.

DEFINITION 1.3. *The boundary operator l is weakly cogent for the interior operator L if there exists a constant C such that*

$$\|u\|_V \leqq C\|Lu\|_V, \qquad u \in C^\infty(\bar{V}), \quad lu = 0 .$$

DEFINITION 1.4. *The boundary operator l_i is locally weakly cogent on S_i for the interior operator L if there exists a constant C such that*

$$\|u\|_V \leqq C\|Lu\|_V, \qquad u \in C_0^\infty(V \cup S_i), \quad l_iu = 0 . \tag{1.4}$$

It is evident that if the boundary operator l is (locally) cogent for the interior operator L (with respect to the norms defined in (1.2) and (1.3)) then the boundary operator l is also (locally) weakly cogent for the interior operator L.

We denote by H the Hilbert space $L^2(V)$ of all square integrable functions in V with norm $\|\cdot\|_V$. We denote by $L_{\min}$ the minimal operator corresponding to $L(D)$, that is, the closure in H of the restriction of L to the set $C_0^\infty(V)$ of functions in $C^\infty(\bar{V})$ which vanish outside compact subsets of V. Further we denote by $\bar{L}$ the formal adjoint of L which corresponds to complex conjugation of the coefficients in L, that is

$$\bar{L} = \bar{L}(D) = \sum_{|\alpha| \leqq n} \bar{L}_\alpha D^\alpha .$$

The maximal operator $L_{\max}$, corresponding to $L(D)$, is then defined as the adjoint in H to the minimal operator corresponding to $\bar{L}(D)$, that is,

$$L_{\max} = (\bar{L}_{\min})^* .$$

We consider a fixed part S_i of S and denote by L_0 the restriction of L to the functions in $C_0^\infty(V \cup S_i)$ with $l_iu=0$. L_0 can then be regarded as an operator from H to H. We consider extensions $\tilde{L}_0$ of L_0 with domain $D(\tilde{L}_0)$ and range $R(\tilde{L}_0)$ such that

$$L_0 \subseteqq \tilde{L}_0 \subseteqq L_{\max} .$$

In this case the equation

$$\tilde{L}_0 u = F, \qquad F \in H , \tag{1.5}$$

can be regarded as an abstract boundary problem, and we introduce after Hadamard the following definition.

DEFINITION 1.5. *We say that the abstract boundary problem* (1.5) *is correctly posed if there exists a constant C such that*

$$\|u\|_V \leqq C \|\tilde{L}_0 u\|_V, \qquad u \in D(\tilde{L}_0) , \tag{1.6}$$

and if the range $R(\tilde{L}_0) = H$.

(Višik [10] calls the operator $\tilde{L}_0$ solvable in this case).

We then have the following theorem which is due to Višik [10].

THEOREM 1.1. *There exists an extension $\tilde{L}_0$ of L_0 corresponding to a correctly posed abstract boundary problem if and only if the boundary operator l_i is locally weakly cogent on S_i for the interior operator L.*

PROOF. It is evident that if there exists an extension $\tilde{L}_0$ of L_0 corresponding to a correctly posed abstract boundary problem, then the boundary operator l_i is locally weakly cogent for L, for (1.6) contains (1.4).

We shall now prove the second part of the theorem. One can show (cf. [4]) that $L_{\max}$ has a bounded right inverse, that is, that there exists a bounded operator R such that $L_{\max} R = I$, the identity operator in H.

Let $\hat{L}_0$ be the closure of L_0 in H. The existence of $\hat{L}_0$ is guaranteed by (1.4). It also follows that the range $R(\hat{L}_0)$ of $\hat{L}_0$ is a closed subspace in H. Let P be the orthogonal projection on the subspace $R(\hat{L}_0)$. Consider the operator T defined by

$$T\varphi = \hat{L}_0^{-1} P\varphi + R(I-P)\varphi, \qquad \varphi \in H .$$

The operator T is defined for all elements in H and is a bounded operator since all operators entering in the definition are bounded. Further we have

$$L_{\max} T\varphi = P\varphi + (I-P)\varphi = \varphi, \qquad \varphi \in H , \tag{1.7}$$

for $L_{\max}$ is an extension of $\hat{L}_0$. It follows from (1.7) that the bounded operator T has an inverse $\tilde{L}_0$, for $T\varphi = 0$ implies $\varphi = 0$. The formula (1.7) also implies that $\tilde{L}_0 \subseteq L_{\max}$. Finally we have that $\tilde{L}_0$ is an extension of L_0 for $(\tilde{L}_0)^{-1} = T$ is an extension of L_0^{-1}. This ends the proof of Theorem 1.1.

In the sequel we specialize the problem and assume that the part S_0 of the boundary, which will be investigated with respect to locally cogent boundary operators, is plane, and that the region V is bounded and situated entirely in one of the halfspaces defined by the plane containing S_0. We suppose, which does not restrict the generality further, that S_0 is a part of the plane $x_0 = 0$, and that V is contained in the set $0 \leqq x_0 \leqq 1$ (see fig. 1). For the sake of convenience we make a minor change in the notations and write $t = x_0$, $x = (x_1, \ldots, x_m)$ and $D_t = D_0$. We denote by S_{t_0} the section $\overline{V} \cap \{t = t_0\}$.

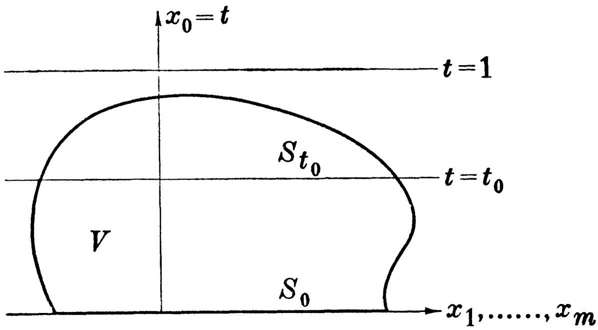

Fig. 1

We shall also assume that the characteristic polynomial $L(\xi, \tau)$ of L,

$$L(\xi, \tau) = \sum_{|\alpha| \leqq n} L_\alpha \tau^{\alpha_0} \xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m} ,$$

is a homogeneous polynomial in $\xi = (\xi_1, \ldots, \xi_m)$ and τ with the coefficient 1 for τ^n. Then $L(\xi, \tau)$ can be factored

$$L(\xi, \tau) = \prod_{i=1}^{n} (\tau - \tau_i(\xi))$$

where $\tau_i(\xi)$ are homogeneous functions of ξ. We finally assume (with a slight change in the above notations) that on S_0 is given a boundary operator $lu = (l_1 u, \ldots, l_\sigma u)$, where

$$l_k = l_k(D), \quad k = 1, \ldots, \sigma ,$$

is a differential operator with constant coefficients, the characteristic polynomial $l_k(\xi, \tau)$ of which is homogeneous of degree $\gamma_k \leqq n - 1$.

In the sequel we shall make use of the following lemma.

LEMMA 1.1. *Let λ be a complex number and let $v(t)$ be a continuously differentiable function of the variable t which vanishes for $t = 1$. Then for $0 \leqq t \leqq 1$ we have $(D_t = -i\, d/dt)$*

$$(1.8) \qquad |v(t)| \leqq |v(0)| + \int_0^1 |(D_t - \lambda) v(t)|\, dt, \qquad \text{all } \lambda ,$$

$$(1.9) \qquad |v(t)| \leqq \int_0^1 |(D_t - \lambda) v(t)|\, dt, \qquad \operatorname{Im} \lambda \leqq 0 .$$

PROOF. We first prove (1.9). Since $v(1)=0$ it follows that

$$v(t) = \int_1^t i e^{i(t-s)\lambda}(D_s-\lambda)v(s)\,ds$$

or

$$|v(t)| \leqq \int_t^1 e^{-(t-s)\operatorname{Im}\lambda}|(D_s-\lambda)v(s)|\,ds .$$

Hence for $\operatorname{Im}\lambda \leqq 0$

$$|v(t)| \leqq \int_0^1 |(D_t-\lambda)v(t)|\,dt .$$

It remains to prove (1.8) for $\operatorname{Im}\lambda > 0$. We have for $\operatorname{Im}\lambda > 0$

$$v(t) = v(0)e^{i\lambda t} + \int_0^t i e^{i\lambda(t-s)}(D_s-\lambda)v(s)\,ds$$

or

$$|v(t)| \leqq |v(0)|e^{-t\operatorname{Im}\lambda} + \int_0^t e^{-(t-s)\operatorname{Im}\lambda}|(D_s-\lambda)v(s)|\,ds .$$

Since $\operatorname{Im}\lambda > 0$ this implies

$$|v(t)| \leqq |v(0)| + \int_0^1 |(D_t-\lambda)v(t)|\,dt .$$

2. Boundary operators of the Dirichlet type. In this section we shall consider a boundary operator

$$lu = (l_1u, \ldots, l_\sigma u) ,$$

where the operator $l_k u$ has the characteristic polynomial

$$l_k(\xi, \tau) = \tau^{k-1}, \qquad k = 1, \ldots, \sigma .$$

The homogeneous interior operator L will have the characteristic polynomial $L(\xi, \tau)$, and the functions $\tau_i(\xi)$, $i=1, \ldots, n$, will be the zeros of $L(\xi, \tau)$ considered as a polynomial in τ for fixed ξ. $p(\xi)$ will denote the number of the $\tau_i(\xi)$ with positive imaginary parts and

$$P = \max_\xi p(\xi).$$

The theorems of this section relate the integers P and σ to the local cogency of l for L with respect to appropriate norms. Theorem 2.1 is concerned with weak local cogency, thus

$$\|u\|_1 = \|u\|_V , \tag{2.1}$$

while Theorem 2.2, which also applies to the inhomogeneous boundary problem, makes use of the norms (2.1) and

$$\|\mathscr{L}u\|_2 = \left\{ \|Lu\|_V{}^2 + \sum_{k=1}^{\sigma} \|l_k u\|_{S_0,\,\sigma-k}{}^2 \right\}^{\frac{1}{2}}. \tag{2.2}$$

THEOREM 2.1. *If the boundary operator*

$$lu = (u, D_t u, \ldots, D_t^{\sigma-1} u)$$

is weakly locally cogent on S_0 for the homogeneous interior operator L, then $P \leqq \sigma$.

PROOF. We prove the theorem by showing that the assumption that there exists a point ξ_0 in which more than σ of the numbers $\tau_i(\xi_0)$, $i=1, \ldots, n$, have positive imaginary parts, leads to a contradiction. We thus assume that $\tau_i(\xi_0)$, $i=1, \ldots, \sigma+1$, have positive imaginary parts. We shall construct a family of functions $u_\varepsilon(x, t) \in C_0^\infty(V \cup S_0)$, which satisfy the boundary conditions on S_0 but for which the ratio

$$\|u_\varepsilon\|_V / \|Lu_\varepsilon\|_V \tag{2.3}$$

is not bounded when $\varepsilon \to 0$.

In the construction of the functions $u_\varepsilon(x, t)$ we have had the opportunity of using ideas from an unpublished manuscript by Hörmander on fundamental solutions with support in a half space. The construction will be carried out in the following way. We shall first construct a function $U(x, t)$ with the properties:

$(U)_1$: $U(x, t)$ is infinitely differentiable and not $\equiv 0$ for $t \geqq 0$.

$(U)_2$: $L(D)\,U(x, t) = 0$ for $t \geqq 0$.

$(U)_3$: $(D_t^{k-1}U)(x, 0) = 0$, $k=1, \ldots, \sigma$.

$(U)_4$: There exists for every non-negative integer k and every a α constant $C_{k,\alpha}$ such that for $t \geqq 0$

$$|r^k D^\alpha U(x, t)| \leqq C_{k,\alpha}, \qquad r = (t^2 + x_1{}^2 + \ldots + x_m{}^2)^{\frac{1}{2}}.$$

We then take a function $\psi(x, t)$ with the following properties:

$(\psi)_1$: $\psi(x, t) \in C_0^\infty(V \cup S_0)$

$(\psi)_2$: $\psi(x, t) \equiv 1$ in a spherical neighbourhood with radius $\varrho > 0$ of the origin.

We put

$$u_\varepsilon(x, t) = \varepsilon^{-(m+1)/2}\, \psi(x, t)\, U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right), \qquad \varepsilon > 0, \tag{2.4}$$

and will presently show that the functions $u_\varepsilon(x, t)$ defined in (2.4) satisfy the following conditions:

$(u_\varepsilon)_1$: $u_\varepsilon \in C_0^\infty(V \cup S_0)$.

$(u_\varepsilon)_2$: $(D_t^{k-1} u_\varepsilon)(x, 0) = 0, \ k = 1, \ldots, \sigma$.

$(u_\varepsilon)_3$: $\|u_\varepsilon\|_V^2 \to \iint\limits_{t \geq 0} |U|^2 \, dx \, dt \neq 0$ when $\varepsilon \to 0$.

$(u_\varepsilon)_4$: $\|Lu_\varepsilon\|_V^2 \to 0$ when $\varepsilon \to 0$.

The properties $(u_\varepsilon)_1, \ldots, (u_\varepsilon)_4$ now show that the ratio (2.3) does not remain bounded when $\varepsilon \to 0$.

We now prove the existence of a function $U(x, t)$ with the properties $(U)_1, \ldots, (U)_4$.

We can suppose that the functions $\tau_i(\xi)$, $i = 1, \ldots, \sigma + 1$, are analytic in ξ and have positive imaginary parts in a neighbourhood Ω' of a point ξ_0'. For since $\sigma + 1$ of the numbers $\tau_i(\xi)$, $i = 1, \ldots, n$, have positive imaginary parts in ξ_0, then this is true also in a neighbourhood Ω of ξ_0. Now, if we factor $L(\xi, \tau)$ into irreducible factors, there is arbitrarily close to ξ_0 a point ξ_0' in Ω, in which all the irreducible factors have simple zeros as polynomials in τ. These zeros are then analytic in a neighbourhood Ω' of ξ_0' with $\bar{\Omega}' \subset \Omega$.

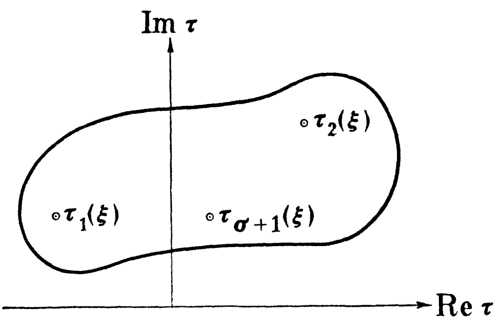

Fig. 2

Let γ be a closed smooth curve in the complex τ-plane which for all $\xi \in \bar{\Omega}'$ encloses the numbers $\tau_i(\xi)$, $i = 1, \ldots, \sigma + 1$, and which has a positive distance to these numbers and to the real axis (see fig. 2).

For $\xi \in \Omega'$ we put

$$L_+(\xi, \tau) = \prod_{j=1}^{\sigma+1} \left(\tau - \tau_j(\xi)\right)$$

and consider the function

$$(2.5) \qquad U(x, t, \xi) = \frac{1}{2\pi i} \int_\gamma \frac{e^{i(x\xi + t\tau)}}{L_+(\xi, \tau)} \, d\tau, \qquad x\xi = x_1\xi_1 + \ldots + x_m\xi_m \, .$$

The function $U(x, t, \xi)$ then has the properties $(U)_1, \ldots, (U)_3$ and a part of the property $(U)_4$. In order to get a function which satisfies also the property $(U)_4$ completely, we take a function $\varphi(\xi)\in C_0^\infty(\Omega')$ with

$$\int \varphi(\xi)\, d\xi = 1$$

and construct

$$U(x, t) = \int U(x, t, \xi)\, \varphi(\xi)\, d\xi\ . \tag{2.6}$$

We shall then prove that the function defined by (2.5) and (2.6) has the properties $(U)_1, \ldots, (U)_4$.

$\underline{(U)_1}$: It is immediately seen that $U(x, t, \xi)$ can be differentiated an arbitrary number of times with respect to x and t under the integral sign. Owing to the choice of Ω', the coefficients of $L_+(\xi, \tau)$ considered as a polynomial in τ are analytic functions of ξ in Ω'. Therefore the function $U(x, t, \xi)$ may also be differentiated an arbitrary number of times under the integral sign with respect to ξ. This will be used in the proof of $(U)_4$. It follows that the function $U(x, t)$ defined in (2.6) is infinitely differentiable, and the differentiation can be performed under the integral sign in (2.6).

In order to see that $U(x, t)$ is not identically zero, we form

$$(D_t^\sigma U)(x, 0, \xi) = e^{ix\xi}\frac{1}{2\pi i}\int_\gamma \frac{\tau^\sigma}{L_+(\xi, \tau)}\, d\tau = e^{ix\xi}\ . \tag{2.7}$$

The last equality in (2.7) follows since γ may be deformed into a circle γ_R with centre at the origin and radius R, and

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\tau^\sigma}{L_+(\xi, \tau)}\, d\tau &= \frac{1}{2\pi i}\int_{\gamma_R} \frac{\tau^\sigma}{L_+(\xi, \tau)}\, d\tau \\ &= \frac{1}{2\pi i}\int_{\gamma_R} \frac{d\tau}{\tau} + O\left(\frac{1}{R}\right) = 1 + O\left(\frac{1}{R}\right) \qquad \text{when} \quad R \to \infty\ .\end{aligned}$$

Then $(D_t^\sigma U)(x, 0)$ is essentially the Fourier transform of $\varphi(\xi)$ and thus cannot vanish identically.

It also follows from (2.7) that the function $U(x, t, \xi)$ does not have the property $(U)_4$.

$\underline{(U)_2}$: We have

$$L(D)\, U(x, t, \xi) = \frac{1}{2\pi i}\int_\gamma \frac{L(\xi, \tau)}{L_+(\xi, \tau)}\, e^{i(x\xi + t\tau)}\, d\tau = 0$$

since the integrand is an analytic function of τ in the whole τ-plane. We therefore also have

$$L(D)\,U(x,t) = \int L(D)\,U(x,t,\xi)\,\varphi(\xi)\,d\xi = 0 .$$

$\underline{(U)_3}$: We have for $k=1,\ldots,\sigma$,

$$(D_t^{k-1}U)(x,0,\xi) = e^{ix\xi}\,\frac{1}{2\pi i}\int_\gamma \frac{\tau^{k-1}}{L_+(\xi,\tau)}\,d\tau .$$

In the same way as in the proof of $(U)_1$ above, we have for $k=1,\ldots,\sigma$,

$$\frac{1}{2\pi i}\int_\gamma \frac{\tau^{k-1}}{L_+(\xi,\tau)}\,d\tau = \frac{1}{2\pi i}\int_{\gamma_R} \frac{\tau^{k-1}}{L_+(\xi,\tau)}\,d\tau \to 0 \qquad \text{when} \qquad R\to\infty .$$

It then also follows for $k=1,\ldots,\sigma$ that

$$(D_t^{k-1}U)(x,0) = \int D_t^{k-1}U(x,0,\xi)\,\varphi(\xi)\,d\xi = 0 .$$

$\underline{(U)_4}$: We have

$$\begin{aligned} D^\alpha U(x,t) &= \int D^\alpha U(x,t,\xi)\,\varphi(\xi)\,d\xi \\ &= \int \frac{1}{2\pi i}\int_\gamma \frac{\tau^{\alpha_0}\xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}\,e^{i(x\xi+t\tau)}}{L_+(\xi,\tau)}\,d\tau\,\varphi(\xi)\,d\xi , \end{aligned}$$

and for an arbitrary non-negative integer k,

$$|t^k D^\alpha U(x,t)| \leqq t^k \int \frac{|\xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}|}{2\pi}\int_\gamma \frac{|\tau|^{\alpha_0}\,e^{-t\,\mathrm{Im}\,\tau}}{|L_+(\xi,\tau)|}\,d\tau\,|\varphi(\xi)|\,d\xi .$$

Since $\mathrm{Im}\,\tau \geqq \varkappa > 0$ on γ for some constant $\varkappa$, we have for fixed k and α

$$|t^k D^\alpha U(x,t)| \leqq C t^k e^{-\varkappa t} \leqq C_1 . \tag{2.8}$$

Further we have for an arbitrary $k \geqq 0$ and $j=1,\ldots,n$,

$$(-ix_j)^k D^\alpha U(x,t) = \int (-ix_j)^k e^{ix\xi}\left\{\frac{1}{2\pi i}\int_\gamma \frac{\tau^{\alpha_0}\xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}}{L_+(\xi,\tau)}\,e^{it\tau}\,d\tau\,\varphi(\xi)\right\}d\xi .$$

It follows from the proof of $(U)_1$ that the function inside the brackets is an infinitely differentiable function of ξ in Ω'. Therefore, after a partial integration we get

$$(-ix_j)^k D^\alpha U(x,t) = \int e^{ix\xi}\,\frac{d^k}{d\xi^k}\left\{\frac{1}{2\pi i}\int_\gamma \frac{\tau^{\alpha_0}\xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}}{L_+(\xi,\tau)}\,e^{it\tau}\,d\tau\,\varphi(\xi)\right\}d\xi .$$

This implies

$$(2.9) \qquad |(-ix_j)^k D^\alpha U(x,t)| \leqq C .$$

From the estimate

$$r^k \leqq C\left\{ |t|^k + \sum_{j=1}^{m} |x_j|^k \right\},$$

we finally obtain from (2.8) and (2.9), $|r^k D^\alpha U(x,t)| \leqq C$, which proves the statement.

We finally prove that the functions $u_\varepsilon(x, t)$ in (2.4) have the properties $(u_\varepsilon)_1, \ldots, (u_\varepsilon)_4$.

$\underline{(u_\varepsilon)_1}$: This property is an immediate consequence of $(U)_1$ and $(\psi)_1$.

$\underline{(u_\varepsilon)_2}$: We have

$$(D_t^{k-1} u_\varepsilon)(x,0) = \varepsilon^{-(m+1)/2} \sum_{j=0}^{k-1} \binom{k-1}{j} (D_t^j \psi)(x,0)\, \varepsilon^{-(k-1-j)} (D_t^{k-1-j} U)\left(\frac{x}{\varepsilon}, 0\right) = 0$$

for $k \leqq \sigma$ according to $(U)_3$.

$\underline{(u_\varepsilon)_3}$: We first observe that

$$\iint\limits_{t \geqq 0} |U(x,t)|^2\, dx\, dt$$

is finite according to $(U)_4$ and $\neq 0$ according to $(U)_1$. We have

$$\|u_\varepsilon\|_V{}^2 = \iint\limits_{t \geqq 0} \varepsilon^{-(m+1)} \left| U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x,t) \right|^2 dx\, dt$$

$$= \iint\limits_{t \geqq 0} |U(x,t)\, \psi(\varepsilon x, \varepsilon t)|^2\, dx\, dt \to \iint\limits_{t \geqq 0} |U(x,t)|^2\, dx\, dt$$

when $\varepsilon \to 0$ according to $(\psi)_2$ and Lebesgue's theorem on dominated convergence.

$\underline{(u_\varepsilon)_4}$: For $r \leqq \varrho$ we have according to $(\psi)_2$

$$u_\varepsilon(x,t) = \varepsilon^{-(m+1)/2}\, U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right).$$

Owing to the homogeneity of the operator L we have therefore for $r \leqq \varrho$

$$L u_\varepsilon = \varepsilon^{-(m+1)/2}\, \varepsilon^{-n}\, (LU)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = 0,$$

and we have

$$\|Lu_\varepsilon\|_V^2 = \iint\limits_{\substack{r \geqq \varrho \\ t \geqq 0}} \varepsilon^{-(m+1)} \left| L\left(\psi(x, t)\, U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)\right)\right|^2 dx\, dt$$

$$= \iint\limits_{\substack{r \geqq \varrho/\varepsilon \\ t \geqq 0}} \varepsilon^{-2n} \left|L\big(\psi(\varepsilon x, \varepsilon t)\; U(x, t)\big)\right|^2 dx\, dt$$

$$\leqq C \iint\limits_{\substack{r \geqq \varrho/\varepsilon \\ t \geqq 0}} \varepsilon^{-2n} \sum_{|\alpha| \leqq n} |D^\alpha U(x, t)|^2\, dx\, dt\,.$$

This last integral, however, converges to 0 when $\varepsilon \to 0$, for according to $(U)_4$ we have

$$|D^\alpha U(x, t)|^2 \leqq C r^{-(m+2+2n)} \leqq C \varrho^{-2n} \varepsilon^{2n} r^{-(m+2)}\,,$$

and the integral

$$\iint\limits_{\substack{r \geqq 1 \\ t \geqq 0}} \frac{dx\, dt}{r^{m+2}}$$

is finite. This ends the proof of Theorem 2.1.

THEOREM 2.2. *If $P \leqq \sigma$, then the boundary operator*

$$lu = (u, D_t u, \ldots, D_t^{\sigma-1} u)$$

is locally cogent on S_0 for L with respect to the norms defined by (2.1) *and* (2.2).

PROOF. We can suppose that $\tau_i(\xi)$, $i = 1, \ldots, p(\xi) \leqq \sigma$, are those of the numbers $\tau_i(\xi)$, $i = 1, \ldots, n$, which have positive imaginary parts.

We denote by $\hat{u}(\xi, t)$ the Fourier transform of $u(x, t) \in C_0^\infty(V \cup S_0)$ with respect to x

$$\hat{u}(\xi, t) = (2\pi)^{-m/2} \int e^{-ix\xi}\, u(x, t)\, dx\,.$$

The function $\hat{u}(\xi, t)$ then satisfies the assumptions of Lemma 1.1, and we obtain from (1.8)

$$|\hat{u}(\xi, t)| \leqq |\hat{u}(\xi, 0)| + \int_0^1 \left|\big(D_t - \tau_1(\xi)\big)\hat{u}(\xi, t)\right| dt\,. \tag{2.10}$$

The same reasoning gives for the function $\big(D_t - \tau_1(\xi)\big)\hat{u}(\xi, t)$,

$$\begin{aligned} &\left|\big(D_t - \tau_1(\xi)\big)\hat{u}(\xi, t)\right| \\ &\quad \leqq \left|\big(D_t - \tau_1(\xi)\big)\hat{u}(\xi, 0)\right| + \int_0^1 \left|\big(D_t - \tau_1(\xi)\big)\big(D_t - \tau_2(\xi)\big)\hat{u}(\xi, t)\right| dt\,. \end{aligned} \tag{2.11}$$

Since for any i, $\tau_i(\xi)$ is a homogeneous function of ξ, we have

$$|\tau_i(\xi)| \leqq C|\xi|, \qquad i = 1, \ldots, n; \qquad |\xi| = (\xi_1^2 + \ldots + \xi_m^2)^{\frac{1}{2}},$$

and we obtain from (2.11)

$$(2.12) \quad \left|\left(D_t - \tau_1(\xi)\right)\hat{u}(\xi, t)\right| \leqq C|\xi|\,|\hat{u}(\xi, 0)| + |(D_t\hat{u})(\xi, 0)| + \int_0^1 \left|\left(D_t - \tau_1(\xi)\right)\left(D_t - \tau_2(\xi)\right)\hat{u}(\xi, t)\right| dt .$$

Together with (2.10), (2.12) gives

$$|\hat{u}(\xi, t)| \leqq C(1 + |\xi|)\,|\hat{u}(\xi, 0)| + |(D_t\hat{u})(\xi, 0)| + \int_0^1 \left|\left(D_t - \tau_1(\xi)\right)\left(D_t - \tau_2(\xi)\right)\hat{u}(\xi, t)\right| dt .$$

We continue by treating the function

$$\left(D_t - \tau_1(\xi)\right)\left(D_t - \tau_2(\xi)\right)\hat{u}(\xi, t)$$

in the same way, and so on. We obtain after σ steps

$$(2.13) \quad |\hat{u}(\xi, t)| \leqq C \sum_{i+j<\sigma} |\xi|^i\,|(D_t^j\hat{u})(\xi, 0)| + \int_0^1 \left|\prod_{i=1}^{\sigma} \left(D_t - \tau_i(\xi)\right)\hat{u}(\xi, t)\right| dt .$$

Since the numbers $\tau_i(\xi)$, $i = \sigma + 1, \ldots, n$, have non-positive imaginary parts, it follows from Lemma 1.1 that

$$(2.14) \quad \int_0^1 \left|\prod_{i=1}^{\sigma} \left(D_t - \tau_i(\xi)\right)\hat{u}(\xi, t)\right| dt$$
$$\leqq \int_0^1 \left|\prod_{i=1}^{\sigma+1} \left(D_t - \tau_i(\xi)\right)\hat{u}(\xi, t)\right| dt$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leqq \int_0^1 \left|\prod_{i=1}^{n} \left(D_t - \tau_i(\xi)\right)\hat{u}(\xi, t)\right| dt$$
$$= \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|\, dt .$$

Together with (2.13), (2.14) gives

$$(2.15)\qquad |\hat{u}(\xi, t)| \leqq C\left\{ \sum_{i+j<\sigma} |\xi|^i\, |(D_t^j \hat{u})(\xi, 0)| + \int_0^1 |L(\xi, D_t)\, \hat{u}(\xi, t)|\, dt \right\}.$$

After squaring, integration with respect to t between 0 and 1, and trivial estimates, (2.15) can be written,

$$\int_0^1 |\hat{u}(\xi, t)|^2\, dt \leqq C\left\{ \sum_{i+j<\sigma} |\xi|^{2i}\, |(D_t^j \hat{u})(\xi, 0)|^2 + \int_0^1 |L(\xi, D_t)\, \hat{u}(\xi, t)|^2\, dt \right\},$$

whence after integration with respect to ξ and use of Parseval's formula,

$$\|u\|_V{}^2 \leqq C\left\{ \sum_{j=0}^{\sigma-1} \|D_t^j u\|_{S_0,\, \sigma-1-j}{}^2 + \|L(D)\, u\|_V{}^2 \right\},$$

or with (2.1) and (2.2),

$$\|u\|_1 \leqq C\, \|\mathscr{L}u\|_2\,,$$

which proves Theorem 2.2.

We illustrate the results obtained by some simple examples.

Example 1. The Cauchy boundary operator $lu=(l_1u, \ldots, l_nu)$ with $l_k(\xi, \tau)=\tau^{k-1}$, $k=1, \ldots, n$, is locally cogent on S_0 for any homogeneous differential operator L of order n.

For the number of zeros of $L(\xi, \tau)$ with positive imaginary parts is trivially $\leqq n$. This is essentially contained in Theorem 0.1.

Example 2. The "empty" boundary operator ($\sigma=0$) is locally cogent on S_0 for a homogeneous differential operator L if and only if L is hyperbolic with respect to the plane $t=0$.

For if $\sigma=0$, according to Theorem 2.1, none of the numbers $\tau_i(\xi)$, $i=1, \ldots, n$, may have a positive imaginary part. Owing to the homogeneity of $\tau_i(\xi)$, this also implies that none of them may have a negative imaginary part. Therefore all zeros of $L(\xi, \tau)$ must be real. The other part of the statement follows immediately from Theorem 2.2.

Example 3. The boundary operator $lu\equiv u\,(\sigma=1)$ is locally cogent on S_0 for the differential operator L with the characteristic polynomial

$$L(\xi, \tau) = \tau^2 + \xi_1{}^2 + \ldots + \xi_p{}^2 - \xi_{p+1}{}^2 - \ldots - \xi_m{}^2, \qquad p > 0\,.$$

In fact, we have $P=1$. This result could be obtained from the proof of a uniqueness theorem of Owens [7].

Remark. In Theorem 2.2 we only estimated $\|u\|_V$. One might expect that it would be possible in the case of locally cogent boundary operators to estimate $\|Mu\|_V$ where M is an arbitrary differential operator with constant coefficients, weaker than L, that is, such that

$$\|Mu\|_V \leqq C\|Lu\|_V, \qquad u \in C_0^\infty(V)\,.$$

Hörmander [4] proved that M is weaker than L if and only if

$$|M(\xi, \tau)|^2 \leqq C \sum_{|\alpha| \geqq 0} |D^\alpha L(\xi, \tau)|^2\,. \tag{2.16}$$

(D^α here acts on $L(\xi, \tau)$ considered as a function of ξ and τ). We shall give two simple examples which prove that such an estimate is in general not possible.

We first consider an operator $L(D)$ which is the product of a hyperbolic operator $H(D)$ of order h and an elliptic operator $E(D)$ of order $n-h$. For the corresponding characteristic polynomials we have

$$L(\xi, \tau) = H(\xi, \tau)\, E(\xi, \tau)\,.$$

The operator $M(D)$ with the characteristic polynomial

$$M(\xi, \tau) = H(\xi, \tau)\, \tau^{n-h}$$

is then weaker than $L(D)$ since (cf. (2.16))

$$|H(\xi, \tau)|^2\, |\tau^{n-h}|^2 \leqq C\,|H(\xi, \tau)|^2\, |E(\xi, \tau)|^2 = C\,|L(\xi, \tau)|^2\,.$$

Suppose that the maximal number of zeros with positive imaginary parts is σ. We shall prove that the estimate

$$\|M(D)u\|_V \leqq C\|L(D)u\|_V \tag{2.17}$$

is not valid for all $u \in C_0^\infty(V \cup S_0)$ with $(D_t^{k-1}u)(x, 0) = 0$, $k = 1, \ldots, \sigma$.

Assume that $\tau_i(\xi_0)$, $i = 1, \ldots, \sigma$, have positive imaginary parts and that $\tau_{\sigma+1}(\xi_0)$ is real. We put

$$L_+(\xi_0, \tau) = \prod_{i=1}^{\sigma+1} \big(\tau - \tau_i(\xi_0)\big)\,,$$

and construct in the same way as in the proof of Theorem 2.1 the function

$$U(x, t) = \frac{e^{ix\xi_0}}{2\pi i} \int_\gamma \frac{e^{it\tau}}{L_+(\xi_0, \tau)}\, d\tau$$

where γ is a curve in the complex τ-plane which encloses the numbers $\tau_i(\xi_0)$, $i = 1, \ldots, \sigma + 1$. $U(x, t)$ has the properties:

$(U)_1$: $U(x, t)$ is infinitely differentiable and $M(D)U(x, t)$ is not $\equiv 0$ for $t \geqq 0$.

$(U)_2$: $L(D)U(x, t)=0$, for $t \geqq 0$.
$(U)_3$: $(D_t^{k-1}U)(x, 0)=0$, $k=1, \ldots, \sigma$.
$(U)_4$: There exists for every α a constant C_α such that for $t \geqq 0$

$$|D^\alpha U(x, t)| \leqq C_\alpha .$$

The properties $(U)_1$, $(U)_2$, $(U)_3$ are proved in the same way as the corresponding properties in the proof of Theorem 2.1. The property $(U)_4$ is weaker in this case because of the fact that $L_+(\xi_0, \tau)$ has a real zero. Since this zero is simple we can write

$$U(x, t) = e^{i(x\xi_0+t\tau_{\sigma+1}(\xi_0))} \prod_{i=1}^{\sigma} \left(\tau_{\sigma+1}(\xi_0)-\tau_i(\xi_0)\right)^{-1} + \\ + \frac{e^{ix\xi_0}}{2\pi i} \int_{\gamma'} \frac{e^{it\tau}\, d\tau}{L_+(\xi_0, \tau)} \tag{2.18}$$

where γ' is a curve enclosing the zeros $\tau_i(\xi_0)$, $i=1, \ldots, \sigma$, but not $\tau_{\sigma+1}(\xi_0)$. The first term on the right hand side in (2.18) evidently has the property $(U)_4$ and the second term can easily be seen to have this property too (cf. the proof of the earlier $(U)_4$).

We then form

$$u_\varepsilon(x, t) = \varepsilon^{n-\frac{1}{2}}\, U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t), \quad \varepsilon > 0 ,$$

where $\psi(x, t)$ is a function in $C_0^\infty(V \cup S_0)$ with $\psi(x, 0)$ not $\equiv 0$. The following properties are then fulfilled for the functions $u_\varepsilon(x, t)$:

$(u_\varepsilon)_1$: $u_\varepsilon(x, t) \in C_0^\infty(V \cup S_0)$.
$(u_\varepsilon)_2$: $(D_t^{k-1}u_\varepsilon)(x, 0)=0$, $k=1, \ldots, \sigma$.
$(u_\varepsilon)_3$: $\|M(D)u_\varepsilon\|_V / \|L(D)u_\varepsilon\|_V \to \infty$ when $\varepsilon \to 0$.

The properties $(u_\varepsilon)_1$ and $(u_\varepsilon)_2$ are proved in exactly the same way as before. According to Leibniz' formula for the differentiation of a product we have

$$M(D)u_\varepsilon(x, t) = \varepsilon^{-\frac{1}{2}}\left(M(D)U\right)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t) + O(\varepsilon^{\frac{1}{2}}) . \tag{2.19}$$

For the first term on the right hand side we obtain

$$\left\| \varepsilon^{-\frac{1}{2}}\left(M(D)U\right)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t)\right\|_V^2 \\ = \iint_{t \geqq 0} \varepsilon^{-1} \left|\left(M(D)U\right)\left(0, \frac{t}{\varepsilon}\right)\right|^2 |\psi(x, t)|^2\, dx\, dt \\ = \iint_{t \geqq 0} \left|\left(M(D)U\right)(0, t)\right|^2 |\psi(x, \varepsilon t)|^2\, dx\, dt . \tag{2.20}$$

Here we have used the fact that $|M(D)U(x,t)| = |(M(D)U)(0,t)|$ which follows from $|e^{ix\xi_0}| = 1$. Fatou's lemma implies that

$$(2.21)\qquad \varliminf_{\varepsilon\to 0} \iint\limits_{t\geqq 0} |(M(D)U)(0,t)|^2\, |\psi(x,\varepsilon t)|^2\, dx\, dt \geqq \int_0^\infty |(M(D)U)(0,t)|^2\, dt \int |\psi(x,0)|^2\, dx \neq 0\,.$$

According to (2.19), (2.20), and (2.21) we have

$$\varliminf_{\varepsilon\to 0} \|M(D)u_\varepsilon\|_V \neq 0\,.$$

Because of $(U)_2$ we have

$$L(D)u_\varepsilon(x,t) = \varepsilon^{-\frac{1}{2}}\left(L(D)U\right)\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\psi(x,t) + O(\varepsilon^{\frac{1}{2}}) = O(\varepsilon^{\frac{1}{2}})\,,$$

and hence

$$\|L(D)u_\varepsilon(x,t)\|_V = O(\varepsilon^{\frac{1}{2}})\,,$$

which establishes $(u_\varepsilon)_3$.

The properties $(u_\varepsilon)_1$, $(u_\varepsilon)_2$, $(u_\varepsilon)_3$ immediately show that the estimate (2.17) is not valid.

In the two-dimensional case every homogeneous differential operator with constant coefficients which is neither hyperbolic nor elliptic is such a product of an elliptic and a hyperbolic operator. For hyperbolic and elliptic operators, on the other hand, it is possible to estimate all weaker operators. For elliptic operators this follows from the papers quoted in the introduction (cf. [1], [8]). For hyperbolic operators with simple characteristics it is a special case of a theorem of Gårding [3]. For hyperbolic operators with multiple characteristics it is not explicitly stated in the literature but is easily proved in the same way as in the proof of Theorem 2.2, since in this case the operators weaker than L are the linear combinations of operators, the characteristic polynomials of which are factors of $L(\xi,\tau)$.

In general it is not even possible to estimate the operators corresponding to derivatives of $L(\xi,\tau)$. In the special case of two dimensions and simple characteristics, however, we shall see in Section 3 (Theorem 3.2), that this is possible. We here give a counter-example in two dimensions and with one double characteristic.

Let L have the characteristic polynomial

$$L(\xi,\tau) = \tau^2(\tau - i\xi)\,.$$

Thus

$$\frac{\partial}{\partial\xi}L(\xi,\tau) = -i\tau^2\,.$$

The maximal number of zeros of $L(\xi, \tau)$ with positive imaginary parts is here one. It could therefore be expected that for all $u(x, t) \in C_0^\infty(V \cup S_0)$ with $u(x, 0) = 0$, the estimate

$$\|D_t^2 u\|_V \leqq C\|Lu\|_V \tag{2.22}$$

would be valid. To prove the contrary we put

$$U_1(x, t) = U_1(x) = \int_{-\infty}^{+\infty} e^{ix\xi} \varphi(\xi)\, d\xi ,$$

$$U_2(x, t) = \int_{-\infty}^{+\infty} e^{ix\xi - t\xi} \varphi(\xi)\, d\xi ,$$

$$U(x, t) = U_1(x, t) - U_2(x, t) ,$$

where $\varphi(\xi)$ is a function with compact support on the positive half-axis and

$$\int_{-\infty}^{+\infty} \varphi(\xi)\, d\xi = 1 .$$

We then put for $\varepsilon > 0$

$$u_{i,\varepsilon}(x, t) = \varepsilon\, U_i\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t), \qquad i = 1, 2 ,$$

$$u_\varepsilon(x, t) = u_{1,\varepsilon}(x, t) - u_{2,\varepsilon}(x, t) ,$$

where $\psi(x, t)$ is a function in $C_0^\infty(V \cup S_0)$ with $\psi(x, t) \equiv 1$ in a neighbourhood of the origin. We then have $u_\varepsilon \in C_0^\infty(V \cup S_0)$ and $u_\varepsilon(x, 0) = 0$. Further

$$D_t^2 u_{1,\varepsilon}(x, t) = \varepsilon\, U_1\left(\frac{x}{\varepsilon}\right) D_t^2 \psi(x, t) \tag{2.23}$$

and

$$D_t^2 u_{2,\varepsilon}(x, t) = \varepsilon^{-1}(D_t^2 U_2)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t) + \tag{2.24}$$

$$+\ 2(D_t U_2)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) D_t \psi(x, t) + \varepsilon\, U_2\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) D_t^2 \psi(x, t) .$$

It easily follows from (2.23) and (2.24) that

$$\lim_{\varepsilon \to 0} \|D_t^2 u_\varepsilon\|_V^2 = \lim_{\varepsilon \to 0} \left\|\varepsilon^{-1} (D_t^2 U_2)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \psi(x, t)\right\|_V^2 \tag{2.25}$$

$$= |\psi(0, 0)|^2 \iint_{t \geqq 0} |D_t^2 U_2(x, t)|^2\, dx\, dt \neq 0 ,$$

for

$$D_t^2 U_2(x, t) = -\int_{-\infty}^{+\infty} \xi^2 e^{ix\xi - t\xi} \varphi(\xi)\, d\xi$$

is not $\equiv 0$. On the other hand it follows from (2.23) that

$$L(D) u_{1,\varepsilon}(x, t) = -i(D_x U_1)\left(\frac{x}{\varepsilon}\right) D_t^2 \psi(x, t) + O(\varepsilon)\,,$$

whence

$$(2.26)\quad \lim_{\varepsilon \to 0} \|L(D) u_{1,\varepsilon}(x, t)\|_V{}^2 = \lim_{\varepsilon \to 0} \iint_{t \geqq 0} \left| (D_x U_1)\left(\frac{x}{\varepsilon}\right) D_t^2 \psi(x, t)\right|^2 dx\, dt$$

$$= \lim_{\varepsilon \to 0} \varepsilon \iint_{t \geqq 0} |D_x U_1(x)\, (D_t^2 \psi)(\varepsilon x, t)|^2\, dx\, dt = 0\,.$$

Further it is easily proved in the same way as in the proof of $(u_\varepsilon)_4$ in the proof of Theorem 2.1 that

$$(2.27)\qquad \lim_{\varepsilon \to 0} \|L(D) u_{2,\varepsilon}\|_V = 0\,.$$

(2.25), (2.26) and (2.27) therefore contradict (2.22) when $\varepsilon \to 0$.

3. The two-dimensional case. In this section we shall consider a differential operator $L(D)$ with the characteristic polynomial

$$(3.1)\qquad L(\xi, \tau) = \prod_{i=1}^{n} (\tau - \tau_i \xi)$$

where τ_i, $i = 1, \ldots, n$, are distinct complex numbers and ξ and τ are complex variables. We denote by $\tau_i{}^+$, $\tau_i{}^-$, and $\tau_i{}^0$, $i = 1, \ldots, \nu^+, \nu^-$, and ν^0, respectively, those of the numbers τ_i, $i = 1, \ldots, n$, which have $\operatorname{Im} \tau_i$ positive, negative, and $= 0$. We also consider a boundary operator $lu = (l_1 u, \ldots, l_\sigma u)$ where $l_k(D)$ has the homogeneous characteristic polynomial $l_k(\xi, \tau)$ with constant coefficients of degree $\gamma_k \leqq n-1$, $k = 1, \ldots, \sigma$. We put

$$(3.2)\qquad l_k(\tau) = l_k(1, \tau)\,.$$

For the statement of the results we introduce the two matrices

$$\Lambda^+ = \begin{pmatrix} l_1(\tau_1{}^+) & \ldots & l_1(\tau_{\nu^+}{}^+) \\ \ldots & \ldots & \ldots \\ l_\sigma(\tau_1{}^+) & & l_\sigma(\tau_{\nu^+}{}^+) \end{pmatrix}$$

and

$$\Lambda^- = \begin{pmatrix} l_1(\tau_1^-) & \ldots & l_1(\tau_{\nu^-}^-) \\ \ldots & \ldots & \ldots \\ l_\sigma(\tau_1^-) & & l_\sigma(\tau_{\nu^-}^-) \end{pmatrix}.$$

As in Section 2, our results are given by two theorems, the first establishing the necessity and the second the sufficiency of our conditions. We first make use of the norms

$$\|u\|_1 = \|u\|_V, \tag{3.3}$$

and

$$\|\mathscr{L}u\|_2 = \left\{ \|Lu\|_V^2 + \sum_{k=1}^{\sigma} \|l_k u\|_{S_0, p_k}^2 \right\}^{\frac{1}{2}}, \tag{3.4}$$

where p_k, $k=1, \ldots, \sigma$, are arbitrary non-negative integers.

Theorem 3.1. *If the boundary operator $lu=(l_1 u, \ldots, l_\sigma u)$ is locally cogent on S_0 for the differential operator L with respect to the norms defined by* (3.3) *and* (3.4), *then the matrices Λ^+ and Λ^- have rank ν^+ and ν^-, respectively. In particular, σ is $\geqq \max(\nu^+, \nu^-)$.*

Proof. This theorem can be proved in the same way that Theorem 2.1 was. We give, however, an alternative proof (cf. Schechter [9]), which in this case becomes simpler. In the proof we shall suppose that the theorem is false and then obtain a contradiction by means of a family of functions $u_\varepsilon(x, t)$ which belong to $C_0^\infty(V \cup S_0)$ but for which the ratio

$$\|u_\varepsilon\|_1 / \|\mathscr{L}u_\varepsilon\|_2 \tag{3.5}$$

is not bounded when $\varepsilon \to 0$.

We thus suppose that the rank of at least one of the matrices Λ^+ and Λ^- is less than ν^+ or ν^-, respectively. We suppose that the rank of Λ^+ is less than ν^+.

For $\varepsilon \geqq 0$ we put

$$U_\varepsilon(x, t) = \sum_{j=1}^{\nu^+} C_j(x+\tau_j^+ t+\varepsilon i)^{-1} \tag{3.6}$$

and show that the constants C_j, $j=1, \ldots, \nu^+$, can be chosen such that $U_\varepsilon(x, t)$ will have the following properties:

$(U_\varepsilon)_1$: $U_\varepsilon(x, t)$ is infinitely differentiable for $t \geqq 0$ and $\varepsilon > 0$.

$(U_\varepsilon)_2$: $U_0(x, t)$ has a singularity at the origin and, for arbitrary $\varrho > 0$,

$$\iint_{x^2+t^2 \leqq \varrho^2} |U_\varepsilon|^2 \, dx \, dt \to \infty \quad \text{when} \quad \varepsilon \to 0.$$

$(U_\varepsilon)_3$: $L(D) U_\varepsilon(x, t) = 0$ for $t \geqq 0$ and $\varepsilon > 0$.

$(U_\varepsilon)_4$: $\big(l_k(D) U_\varepsilon\big)(x, 0) = 0$ for $k=1, \ldots, \sigma$ and $\varepsilon > 0$.

If $\psi(x, t)$ is a function satisfying the conditions

$(\psi)_1$: $\psi(x, t) \in C_0^\infty(V \cup S_0)$
$(\psi)_2$: $\psi(x, t) \equiv 1$ for all (x, t) with $r \leqq \varrho_0$ where ϱ_0 is some number > 0,

then

$$u_\varepsilon(x, t) = \psi(x, t)\, U_\varepsilon(x, t) \tag{3.7}$$

has the properties

$(u_\varepsilon)_1$: $u_\varepsilon \in C_0^\infty(V \cup S_0)$ for $\varepsilon > 0$.
$(u_\varepsilon)_2$: $\|u_\varepsilon\|_1 \to \infty$ when $\varepsilon \to 0$.
$(u_\varepsilon)_3$: $\|Lu_\varepsilon\|_V$ is bounded when $\varepsilon \to 0$.
$(u_\varepsilon)_4$: $\|l_k u_\varepsilon\|_{S_0, p_k}$ is bounded when $\varepsilon \to 0$.

From the properties $(u_\varepsilon)_1, \ldots, (u_\varepsilon)_4$ it follows that the functions $u_\varepsilon(x, t)$ defined in (3.7) contradict (3.5) for $\varepsilon \to 0$.

We now prove that C_j, $j = 1, \ldots, \nu^+$, in (3.6) can be chosen such that $(U_\varepsilon)_1, \ldots, (U_\varepsilon)_4$ are satisfied.

$\underline{(U_\varepsilon)_1}$: This condition is satisfied for every choice of C_j, $j = 1, \ldots, \nu^+$, for $U_\varepsilon(x, t)$ has singularities only when

$$x + \tau_j^+ t + \varepsilon i = 0, \qquad j = 1, \ldots, \nu^+,$$

that is, at the points

$$\left(\frac{\operatorname{Re} \tau_j^+}{\operatorname{Im} \tau_j^+}\varepsilon, \; -\frac{\varepsilon}{\operatorname{Im} \tau_j^+}\right), \qquad j = 1, \ldots, \nu^+, \tag{3.8}$$

which are all situated in the region $t < 0$ if $\varepsilon > 0$.

$\underline{(U_\varepsilon)_2}$: For $\varepsilon = 0$, all these singularities are situated at the origin. We have

$$U_0(x, t) = \sum_{j=1}^{\nu^+} C_j (x + \tau_j^+ t)^{-1}.$$

If C_j, $j = 1, \ldots, \nu^+$, are not all $= 0$, $U_0(x, t)$ is not $\equiv 0$, which is easily seen since $\tau_j^+ \neq \tau_k^+$ for $j \neq k$. We see if we introduce polar coordinates and use the homogeneity of $U_0(x, t)$ that for arbitrary $\varrho > 0$,

$$\iint\limits_{\substack{x^2+t^2 \leqq \varrho^2 \\ t \geqq 0}} |U_0(x, t)|^2 \, dx\, dt = \int_0^\varrho \frac{dr}{r} \int_0^\pi |U_0(\cos\varphi, \sin\varphi)|^2 \, d\varphi$$

is infinite. According to Fatou's lemma we have

$$\varliminf_{\varepsilon \to 0} \iint\limits_{\substack{x^2+t^2 \leqq \varrho^2 \\ t \geqq 0}} |U_\varepsilon(x, t)|^2 \, dx\, dt \geqq \iint\limits_{\substack{x^2+t^2 \leqq \varrho^2 \\ t \geqq 0}} |U_0(x, t)|^2 \, dx\, dt = +\infty.$$

which proves the assertion.

$\underline{(U_\varepsilon)_3}$: We have

$$L(D)\big((x+\tau_j{}^+t+\varepsilon i)^{-1}\big) = L(1, \tau_j{}^+)\,(-1)^n\, n!\,(x+\tau_j{}^+t+\varepsilon i)^{-(n+1)} = 0$$

since $L(1, \tau_j{}^+)=0$. Therefore we have according to the linearity of $L(D)$

$$L(D)\,U_\varepsilon(x, t) = 0 .$$

$\underline{(U_\varepsilon)_4}$: We have in the same way as above

$$l_k(D)\big((x+\tau_j{}^+t+\varepsilon i)^{-1}\big) = l_k(\tau_j{}^+)(-1)^{\gamma_k}\,\gamma_k!\,(x+\tau_j{}^+t+\varepsilon i)^{-(\gamma_k+1)}, \qquad k = 1, \ldots, \sigma ,$$

whence

$$(3.9) \qquad \big(l_k(D)U_\varepsilon\big)(x, 0) = (-1)^{\gamma_k}\,\gamma_k!\,(x+\varepsilon i)^{-(\gamma_k+1)} \sum_{j=1}^{\nu+} C_j l_k(\tau_j{}^+), \qquad k = 1, \ldots, \sigma .$$

Since by assumption the rank of Λ^+ is less than ν^+, we can find C_j, $j=1, \ldots, \nu^+$, not all zero, such that

$$\sum_{j=1}^{\nu+} C_j l_k(\tau_j{}^+) = 0, \qquad k = 1, \ldots, \sigma .$$

With this choice of C_j, $j=1, \ldots, \nu^+$, we have according to (3.9)

$$\big(l_k(D)U_\varepsilon\big)(x, 0) = 0, \qquad k = 1, \ldots, \sigma .$$

In the sequel we assume C_j, $j=1, \ldots, \nu^+$, fixed in this way.

We finally prove that the functions $u_\varepsilon(x, t)$ have the properties $(u_\varepsilon)_1, \ldots, (u_\varepsilon)_4$.

$\underline{(u_\varepsilon)_1}$: This is an immediate consequence of the properties $(U_\varepsilon)_1$ and $(\psi)_1$.

$\underline{(u_\varepsilon)_2}$: We have for a suitable $\varrho > 0$ because of $(\psi)_2$

$$\|u_\varepsilon\|_1{}^2 \geqq \iint\limits_{\substack{r \leqq \varrho \\ t \geqq 0}} |U_\varepsilon(x, t)|^2\, dx\, dt \to \infty \quad \text{when} \quad \varepsilon \to 0$$

according to $(U_\varepsilon)_2$.

$\underline{(u_\varepsilon)_3}$: According to (3.7), $(\psi)_2$, and $(U_\varepsilon)_3$, $L(D)u_\varepsilon(x, t)=0$ for all points (x, t) with $t \geqq 0$ and $r=(x^2+t^2)^{\frac{1}{2}} \leqq \varrho_0$. We now only consider so small ε that all singularities (3.8) of $U_\varepsilon(x, t)$ have distances $\leqq \frac{1}{2}\varrho_0$ to the origin. Since the singularities of $u_\varepsilon(x, t)$ for all ε considered then have distances $\geqq \frac{1}{2}\varrho_0$ to the set $r \geqq \varrho_0$ also, we realize that for $r \geqq \varrho_0$ and thus for all $t \geqq 0$,

$$(3.10) \qquad |L(D)\,u_\varepsilon(x, t)| \leqq C .$$

The constant C in (3.10) is independent of ε. This proves the assertion.

$\underline{(u_\varepsilon)_4}$: The same reasoning can immediately be applied to $l_k u_\varepsilon(x, t)$, $k = 1, \ldots, \sigma$, and their derivatives with respect to x, which establishes this property.

The case where the rank of Λ^- is less than ν^- is treated in the same way. In this case we replace $\tau_j^+, j = 1, \ldots, \nu^+$ and ε in (3.6) by $\tau_j^-, j = 1, \ldots, \nu^-$ and $-\varepsilon$.

For our second theorem we introduce the norms

$$\|u\|_1 = \|u\|_{V, n-1} \tag{3.11}$$

and

$$\|\mathscr{L}u\|_2 = \left\{ \|Lu\|_V^2 + \sum_{k=1}^{\sigma} \|l_k u\|_{S_0, n-1-\gamma_k}^2 \right\}^{\frac{1}{2}}. \tag{3.12}$$

THEOREM 3.2. *If the matrices Λ^+ and Λ^- have the rank ν^+ and ν^-, respectively, then the boundary operator $lu = (l_1 u, \ldots, l_\sigma u)$ is locally cogent on S_0 for the differential operator L with respect to the norms defined in* (3.11) *and* (3.12).

PROOF. We first introduce some notations. Let $L_i(D)$, $L_i^+(D)$, $L_i^-(D)$ and $L_i^0(D)$ be the differential operators with the characteristic polynomials

$$L_i(\xi, \tau) = \frac{L(\xi, \tau)}{\tau - \tau_i \xi}, \qquad L_i^+(\xi, \tau) = \frac{L(\xi, \tau)}{\tau - \tau_i^+ \xi},$$

$$L_i^-(\xi, \tau) = \frac{L(\xi, \tau)}{\tau - \tau_i^- \xi}, \qquad L_i^0(\xi, \tau) = \frac{L(\xi, \tau)}{\tau - \tau_i^0 \xi}.$$

As in (3.2) we put

$$L_i(\tau) = L_i(1, \tau), \qquad L_i^+(\tau) = L_i^+(1, \tau),$$

$$L_i^-(\tau) = L_i^-(1, \tau), \qquad L_i^0(\tau) = L_i^0(1, \tau).$$

Further, let

$$l_{k, n-1}(\xi, \tau) = \xi^{n-1-\gamma_k} \, l_k(\xi, \tau),$$

that is a differential operator of order $n - 1$ obtained by differentiation of $l_k(D)$ with respect to x.

Let $u(x, t)$ be an arbitrary function $\in C_0^\infty(V \cup S_0)$. We denote by $\hat{u}(\xi, t)$ the Fourier transform of $u(x, t)$ with respect to x.

Consider an arbitrary $\xi \geqq 0$. With $\lambda = \tau_i \xi$, Lemma 1.1 gives

$$|L_i^+(\xi, D_t)\hat{u}(\xi, t)| \leqq \left|(L_i^+(\xi, D_t)\hat{u})(\xi, 0)\right| + \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)| \, dt \tag{3.13}$$

and

$$(3.14)\qquad |L_i^-(\xi, D_t)\hat{u}(\xi, t)| \leqq \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|\, dt$$

and

$$(3.15)\qquad |L_i^0(\xi, D_t)\hat{u}(\xi, t)| \leqq \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|\, dt$$

since $\operatorname{Im} \tau_i^- \xi \leqq 0$ and $\operatorname{Im} \tau_i^0 \xi \leqq 0$.

Now suppose that the rank of Λ^+ is ν^+. Without any loss of generality we can then suppose

$$(3.16)\qquad \begin{vmatrix} l_1(\tau_1^+) & \dots & l_1(\tau_{\nu^+}^+) \\ \dots & \dots & \dots \\ l_{\nu^+}(\tau_1^+) & & l_{\nu^+}(\tau_{\nu^+}^+) \end{vmatrix} \neq 0 .$$

According to Lagrange's interpolation formula, we have

$$(3.17)\qquad l_{k,n-1}(\xi, \tau) = \sum_{j=1}^{n} \frac{l_k(\tau_j)}{L_j(\tau_j)} L_j(\xi, \tau), \qquad k = 1, \dots, \nu^+ .$$

Here $L_j(\tau_j) \neq 0$ since $\tau_i \neq \tau_j$ for $i \neq j$.

According to (3.16), (3.17) can be solved with respect to $L_j^+(\xi, \tau)$, $j = 1, \dots, \nu^+$, and we get with constants a_{jk}, b_{jk}^-, b_{jk}^0

$$L_j^+(\xi, \tau) = \sum_{k=1}^{\nu^+} a_{jk} l_{k,n-1}(\xi, \tau) + \sum_{k=1}^{\nu^-} b_{jk}^- L_k^-(\xi, \tau) + \sum_{k=1}^{\nu^0} b_{jk}^0 L_k^0(\xi, \tau) , \qquad j = 1, \dots, \nu^+ ,$$

whence

$$\left|\left(L_j^+(\xi, D_t)\hat{u}\right)(\xi, 0)\right| \leqq \sum_{k=1}^{\nu^+} |a_{jk}| \left|\left(l_{k,n-1}(\xi, D_t)\hat{u}\right)(\xi, 0)\right| + \sum_{k=1}^{\nu^-} |b_{jk}^-| \left|\left(L_k^-(\xi, D_t)\hat{u}\right)(\xi, 0)\right| + \sum_{k=1}^{\nu^0} |b_{jk}^0| \left|\left(L_k^0(\xi, D_t)\hat{u}\right)(\xi, 0)\right| , \qquad j = 1, \dots, \nu^+ ,$$

and together with (3.14) and (3.15),

$$\left|\left(L_j^+(\xi, D_t)\hat{u}\right)(\xi, 0)\right| \leqq C\left\{ \sum_{i=1}^{\nu^+} \left|\left(l_{k,n-1}(\xi, D_t)\hat{u}\right)(\xi, 0)\right| + \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|\, dt \right\}, \qquad j = 1, \dots, \nu^+ .$$

Combined with (3.13), (3.14), and (3.15) this implies

$$|L_j(\xi, D_t)\hat{u}(\xi, t)| \leqq C\left\{ \sum_{k=1}^{\nu^+} |\xi|^{n-1-\gamma_k} \left|\left(l_k(\xi, D_t)\hat{u}\right)(\xi, 0)\right| + \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|\, dt \right\}, \qquad j = 1, \dots, n ,$$

whence with a new constant C after taking squares and using Cauchy-Schwarz' inequality and possibly after addition of positive terms on the right hand side,

$$|L_j(\xi, D_t)\hat{u}(\xi, t)|^2 \leqq C\left\{\sum_{k=1}^{\sigma} |\xi|^{2(n-1-\gamma_k)} \left|\big(l_k(\xi, D_t)\hat{u}\big)(\xi, 0)\right|^2 + \right. \\ \left. + \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|^2\, dt\right\}, \qquad j = 1, \ldots, n\,,$$

or, since the $L_j(\xi, \tau)$, $j=1, \ldots, n$, constitute a basis for the set of homogeneous polynomials in ξ and τ of degree $n-1$,

$$\sum_{j=0}^{n-1} |\xi^j D_t^{n-1-j}\hat{u}(\xi, t)|^2 \leqq C\left\{\sum_{k=1}^{\sigma} |\xi|^{2(n-1-\gamma_k)} \left|\big(l_k(\xi, D_t)\hat{u}\big)(\xi, 0)\right|^2 + \right. \\ \left. + \int_0^1 |L(\xi, D_t)\hat{u}(\xi, t)|^2\, dt\right\}. \tag{3.18}$$

The inequality (3.18) which is only proved for $\xi \geqq 0$ can with trivial modifications be proved for $\xi \leqq 0$. In this case we use that the rank of Λ^- is ν^-. The inequality (3.18) is therefore valid for all real ξ.

Integration with respect to ξ of (3.18) and application of Parseval's formula gives

$$\sum_{|\alpha|=n-1} \|D^\alpha u\|_{S_t}^2 \leqq C\left\{\sum_{k=1}^{\sigma} \|l_k(D)u\|_{S_0,\, n-1-\gamma_k}^2 + \|L(D)u\|_V^2\right\}. \tag{3.19}$$

Integration with respect to t between 0 and 1 gives

$$\sum_{|\alpha|=n-1} \|D^\alpha u\|_V^2 \leqq C\left\{\sum_{k=1}^{\sigma} \|l_k(D)u\|_{S_0,\, n-1-\gamma_k}^2 + \|L(D)u\|_V^2\right\}. \tag{3.20}$$

Since V is bounded in the direction of the t-axis, we have

$$\sum_{|\alpha|\leqq n-1} \|D^\alpha u\|_V^2 \leqq C \sum_{|\alpha|=n-1} \|D^\alpha u\|_V^2\,. \tag{3.21}$$

Combined with (3.20), (3.21) gives

$$\|u\|_1 \leqq C\|\mathscr{L}u\|_2\,,$$

which completes the proof of Theorem 3.2.

Remark. It follows from (3.19) that Theorem 3.2 remains true with the stronger norm

$$\|u\|_1 = \left\{\sup_{0\leqq t\leqq 1} \sum_{|\alpha|\leqq n-1} \|D^\alpha u\|_{S_t}^2\right\}^{\frac{1}{2}}.$$

EXAMPLE. The boundary operator $lu=(l_1u, \ldots, l_\sigma u)$ with

$$\sigma = \max(\nu^+, \nu^-)$$

and

$$l_k u = D_t^{p+k-1}u, \qquad k = 1, \ldots, \sigma,$$

where p is a fixed integer with $0 \leqq p \leqq n-\sigma$ is locally cogent on S_0 for every $L(D)$ with the characteristic polynomial (3.1).

For the rank of Λ^+ is ν^+ since the determinant

$$\begin{vmatrix} (\tau_1^+)^p \ldots\ldots & (\tau_{\nu^+}^+)^p \\ \ldots\ldots\ldots\ldots & \ldots\ldots \\ (\tau_1^+)^{p+\nu^+-1} & (\tau_{\nu^+}^+)^{p+\nu^+-1} \end{vmatrix} = \prod_{i=1}^{\nu^+} (\tau_i^+)^p \prod_{j>i} (\tau_j^+ - \tau_i^+) \neq 0$$

since according to the assumption

$$\tau_i^+ \neq \tau_j^+ \quad \text{for} \quad i \neq j \quad \text{and} \quad \tau_j^+ \neq 0, \quad j = 1, \ldots, \nu^+.$$

Correspondingly we see that the rank of Λ^- is ν^-.

BIBLIOGRAPHY

1. S. Agmon, *The coerciveness problem for integro-differential forms*, J. Analyse Math. 6 (1958), 183–223.
2. N. Aronszajn, *On coercive integro-differential quadratic forms*, Proc. Conf. Partial Differential Equations, Technical Report No. 14, University of Kansas, 1954, 94–106.
3. L. Gårding, *Solution directe du problème de Cauchy pour les équations hyperboliques*, Colloques internationaux du C. N. R. S. LXXI (Nancy, 9–15 Avril 1956, La théorie des équations aux dérivées partielles), Paris, 1956, 71–90.
4. L. Hörmander, *On the theory of general partial differential operators*, Acta Math. 94 (1955), 161–248.
5. L. Hörmander, *On the regularity of the solutions of boundary problems*, Acta Math. 99 (1958), 225–264.
6. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier Grenoble 6 (1955–1956), 271–355.
7. O. G. Owens, *Uniqueness of solutions of ultrahyperbolic partial differential equations*, Amer. J. Math. 69 (1947), 184–188.
8. M. Schechter, *On estimating elliptic partial differential operators in the L_2 norm*, Amer. J. Math. 79 (1957), 431–443.
9. M. Schechter, *Coerciveness of linear partial differential operators for functions satisfying zero Dirichlet-type boundary data*, Comm. Pure Appl. Math. 11 (1958), 153–174.
10. M. I. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obšč. 1 (1952), 187–246. (Russian.)

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