A NEW DERIVATION OF THE INFORMATION FUNCTION

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The purpose of this note is to prove the following

**Theorem.** Let a function, $H$, satisfy the conditions

(i) $H$ is defined for any set of non-negative arguments with sum 1, and it is symmetric in all arguments.

(ii) $H(x_1, x_2, \ldots, x_{n-1}, u, v) = H(x_1, x_2, \ldots, x_{n}) + x_n H\left(\frac{u}{x_n}, \frac{v}{x_n}\right)$, whenever all terms of the equation have a meaning.

(iii) $H(x, 1-x)$ is integrable, in the sense of Lebesgue, on the interval $0 \leq x \leq 1$.

Then $H$ is determined up to a multiplicative constant.

Weaker forms of this theorem have been proved by Fadiev [1] and Khintchine [2]. They both assume, beside conditions (i) and (ii), the continuity of $H(x, 1-x)$. In addition, Khintchine assumes that

$$H(x_1, x_2, \ldots, x_n) \leq H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

In Shannon and Weaver [3] can be found a simple derivation of the form of $H$, the assumptions being those of Fadiev, and further that

$$H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$$

is an increasing function of $n$.

If my weakening of the conditions is insignificant from an information-theoretic point of view, I do not think that it is so from a purely mathematical one.

The proof of the theorem is direct, by deducing the form of $H$: Conditions (i) and (ii) give

$$H(x, u, v) = H(x, u+v) + (u+v) H\left(\frac{u}{u+v}, \frac{v}{u+v}\right)$$

$$\quad = H(u, x+v) + (x+v) H\left(\frac{x}{x+v}, \frac{v}{x+v}\right),$$

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for $0 \leq x < 1$, $0 \leq u < 1$, $x + u \leq 1$. With $f(x) = H(x, 1 - x)$, (1) gives the functional equation

$$
(2) \quad f(x) + (1 - x) f\left(\frac{u}{1 - x}\right) = f(u) + (1 - u) f\left(\frac{x}{1 - u}\right).
$$

Condition (iii) allows us to integrate (2) with respect to $u$ between the limits 0 and $1 - x$, and also to perform an appropriate change of variable in two of the integrals. The result is:

$$
(3) \quad (1 - x)f(x) + (1 - x)^2 \int_0^1 f(t)dt = \int_0^{1-x} f(t)dt + x^2 \int_x^1 t^{-3}f(t)dt.
$$

Condition (iii) assures the continuity in $x$ of all terms of this equation, except the first one, for $0 < x < 1$. We conclude that $f(x)$ is continuous, and then, by an analogous argument, differentiable for $0 < x < 1$. Upon differentiation, (3) yields

$$
(4) \quad (1 - x)f'(x) - f(x) - 2(1 - x)\int_0^1 f(t)dt = -f(1 - x) - 2x \int_x^1 t^{-3}f(t)dt - x^{-1}f(x).
$$

Note that $f(1 - x)$ cancels against $f(x)$, by condition (i). Then (4) shows the existence of $f''(x)$, and by differentiating (4) and then eliminating $\int_x^1 t^{-3}f(t)dt$, one gets

$$
(5) \quad f''(x) = -2x^{-1}(1-x)^{-1}\int_0^1 f(t)dt,
$$

whence

$$
(6) \quad f(x) = ax + b - 2[x \log x + (1-x) \log (1-x)]\int_0^1 f(t)dt.
$$

Symmetry shows that $a = 0$, and integration from 0 to 1 then gives $b = 0$.

Finally, one finds that $f(0) = f(1) = 0$ by letting $u = 1 - x$ in equation (2), and (6) is seen to yield the general form of $H(x_1, x_2)$. By induction and use of conditions (i) and (ii), (6) is immediately extended to

$$
(7) \quad H(x_1, x_2, \ldots, x_n) = c (x_1 \log x_1 + \ldots + x_n \log x_n).
$$

REFERENCES


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