AN INTEGRAL FORMULA

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1. Introduction. The present note is concerned with a formula for the mean value of the integrals of a function over all \(k\)-dimensional linear varieties passing through the origin in Euclidean \(n\)-dimensional space \(R^n\) \((k=1, 2, \ldots, n-1)\). This mean value refers to a certain invariant measure \(\mu\) on the space \(A^k\) of all such varieties. The space \(A^k\) becomes a metric space (in particular a topological space) when the distance \([\alpha, \beta]\) between two varieties \(\alpha\) and \(\beta\) from \(A^k\) is defined as the Fréchet distance\(^1\) \(\sigma(\Omega_\alpha, \Omega_\beta)\) between the unit spheres \(\Omega_\alpha\) and \(\Omega_\beta\) in \(\alpha\) and \(\beta\):

\[
[\alpha, \beta] = \max\{q(\Omega_\alpha, \Omega_\beta), q(\Omega_\beta, \Omega_\alpha)\},
\]

where

\[
q(\Omega_\alpha, \Omega_\beta) = \sup_{x \in \Omega_\alpha} \inf_{y \in \Omega_\beta} |x - y|.
\]

It is known from integral geometry that there is a unique invariant normalized measure \(\mu \geq 0\) on this topological space \(A^k\). (The invariance refers to the group \(G=O(n)\) of all rotations about the origin \(0 \in R^n\). Normalization means that the total mass \(\mu(A^k)\) equals 1.) This measure \(\mu\) is known explicitly; it has a known density\(^2\) relative to a suitable parametrization of \(A^k\). For the purpose of the present note, the explicit determination of \(\mu\) is unnecessary. The mere existence and uniqueness of such a measure will suffice, and they will follow from a known theorem on invariant integration in homogeneous spaces.

The integral formula in question may be stated as follows:

\[
\int_{A^k} F(\alpha) \, d\mu(\alpha) = \frac{\omega_k}{\omega_n} \int_{R^n} |x|^{k-n} f(x) \, dx.
\]

Here \(f=f(x)\) is an arbitrary Baire function on \(R^n\) with values

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\(^1\) Cf. e. g. N. Bourbaki [2, Chap. 9, exerc. 7, p. 29]. Actually, \(q(\Omega_\alpha, \Omega_\beta) = q(\Omega_\beta, \Omega_\alpha)\) since there always exists an involutory orthogonal transformation in \(R^n\) which interchanges \(\alpha\) and \(\beta\). The supremum and infimum may, of course, be replaced by maximum and minimum, respectively, on account of the compactness of \(\Omega_\alpha\) and \(\Omega_\beta\).

\(^2\) Cf. W. Blaschke [1]. The first determination of \(\mu\) is due to G. Herglotz [6]. For a general exposition of integral geometry, see L. A. Santaló [10].
\[ 0 \leq f(x) \leq +\infty. \]

The corresponding function \( F = F(\alpha) \) is defined on \( A^k \) as follows:

\[
F(\alpha) = \int f(x) \, d\sigma(x),
\]

where \( \sigma \) denotes \( k \)-dimensional Lebesgue measure in \( x \in A^k \). Moreover, \( dx \) refers to \( n \)-dimensional Lebesgue measure, and \(|x|\) denotes the Euclidean length of the vector \( x \) (= the distance between the origin \( 0 \) and the point \( x \)). Finally, \( \omega_m \) denotes the surface of the unit sphere in \( \mathbb{R}^m \), \( m = 1, 2, \ldots \):

\[
\omega_m = \frac{2\pi \cdot m!}{\Gamma(\frac{1}{2} m)}. \]

2. **Invariant integration in homogeneous spaces.** This subject is treated systematically in Weil [11, § 9]. For the sake of convenience we bring here the very simple arguments by which the question of invariant integration in a homogeneous space \( \Gamma = G/H \), where \( G \) is a compact group and \( H \) a closed subgroup of \( G \), is reduced to the well-known special case of invariant integration in a compact group (Haar measure). The homogeneous space \( \Gamma = G/H \) is defined as the space of all left cosets \( \gamma = gH \) of \( G \) modulo \( H \) (\( g \in G \)), the topology on \( \Gamma \) being the finest topology such that the canonical mapping \( \varphi \) of \( G \) onto \( \Gamma \) is continuous (cf. N. Bourbaki [2, Chap. 3, p. 18]). The canonical mapping \( \varphi \) is defined by \( \varphi(g) = gH \). It is an open mapping, i.e. carries open subsets of \( G \) into open subsets of \( \Gamma \) (cf. N. Bourbaki [2, Chap. 3, Prop. 14, p. 18]). The group \( G \) acts on the homogeneous space \( \Gamma \) as a transitive transformation group as follows: If \( g \in G \) and \( \gamma = g_1 H \), then \( g\gamma = gg_1 H \). Since \( G \) was supposed to be compact, the continuous image \( \Gamma = \varphi(G) \) is likewise compact.

Now let \( \lambda \) denote Haar measure on the compact group \( G \), normalized so that \( \lambda(G) = 1 \), and define \( \mu \) as the image of \( \lambda \) under the continuous mapping \( \varphi \) of \( G \) onto \( \Gamma \) (N. Bourbaki [3, § 6, n° 1; in particular Remark 1]). Explicitly,

\[
\mu(E) = \lambda(\varphi^{-1}(E))
\]

for any Borel set \( E \subset \Gamma \). Or, on integral form,

\[
\int_{\Gamma} f(\gamma) \, d\mu(\gamma) = \int_{\Gamma} f(gH) \, d\lambda(g)
\]

for any Baire function \( f \) on \( \Gamma \) with values \( 0 \leq f(\gamma) \leq +\infty \) (N. Bourbaki [3, Prop. 2, p. 74]). Observe that, since \( \varphi \) is open, a set \( E \subset \Gamma \) is a Borel subset of \( \Gamma \) if (and only if) \( \varphi^{-1}(E) \) is a Borel subset of \( G \); and similarly a
function $f$ on $\Gamma$ is a Baire function on $\Gamma$ if (and only if) $f(gH)$ determines a Baire function of $g$ on $G$.

It is immediately verified that $\mu$ is normalized and invariant under $G$. Conversely, if $\mu$ denotes any normalized, $G$-invariant measure on $\Gamma$, then $\mu$ satisfies (4) and hence coincides with the above measure $\mu$. In fact, the invariance of $\mu$ may be expressed as follows:

$$\int_{\Gamma} f(\gamma) \, d\mu(\gamma) = \int_{\Gamma} f(g\gamma) \, d\mu(\gamma)$$

for any $g \in G$ and any function $f$ of the type specified above. Integrating with respect to $d\lambda(g)$, and applying Fubini’s theorem, we obtain

$$\int_{\Gamma} f(\gamma) \, d\mu(\gamma) = \int_{\Gamma} d\mu(\gamma) \int_{G} f(g\gamma) \, d\lambda(g).$$

Each $\gamma \in \Gamma$ has the form $\gamma = kH$ for some $k \in G$, and hence

$$\int_{G} f(g\gamma) \, d\lambda(g) = \int_{G} f(gkH) \, d\lambda(g) = \int_{G} f(gH) \, d\lambda(g)$$

in view of the right invariance of Haar measure on $G$ (applied to the Baire function $f(gH)$ of $g \in G$). Thus $\gamma$ does not enter, and since $\mu$ is normalized, we are led to the equation (4), q.e.d.

3. Interpretation of $A^k$ as a homogeneous space. From now on we take for $G$ the orthogonal group $O(n)$, the elements of which are all orthogonal matrices of order $n$, viewed also as rotations about the origin $0$ in $\mathbb{R}^n$. In particular, $G$ acts on $A^k$ as a transitive transformation group in the obvious way. The usual topology on $G = O(n)$ can be defined by various well-known metrics, for example by taking for the distance between $g_1 \in G$ and $g_2 \in G$ the “operator norm” of the matrix difference $g_1 - g_2$:

$$(5) \quad ||g_1 - g_2|| = \max_{\xi \in \Omega} |g_1 \xi - g_2 \xi|.$$  

(We denote by $\Omega$ the unit sphere in $\mathbb{R}^n$, $\Omega = \{\xi \in \mathbb{R}^n | ||\xi|| = 1\}$.) It is well known that $G$ is compact (cf. e.g. C. Chevalley [4, Theorem 1, p. 4]). For fixed $x \in \mathbb{R}^n$, the mapping $g \rightarrow gx$ of $G$ into $\mathbb{R}^n$ is clearly continuous, and hence the scalar product $\langle gx, y \rangle$ is a continuous function of $g \in G$ for fixed $x$ and $y$ in $\mathbb{R}^n$.

Next, let $\alpha_0 \in A^k$ denote a fixed $k$-dimensional subspace of $\mathbb{R}^n$, and denote by $H$ the subgroup of $G$ consisting of all rotations $g \in G$ which leave $\alpha_0$ invariant (not necessarily point-wise):
\( g_{\alpha_0} = \alpha_0 \) for \( g \in H \).

This amounts to the requirement that \( \langle gx, y \rangle = 0 \) for every pair \((x, y)\) with \(x \in \alpha_0\) and \(y \perp \alpha_0\). Hence \( H \) is closed. Consider now the homogeneous space \( \Gamma = G/H \), and define a correspondence \( \psi \) between \( \Gamma \) and \( A^k \) as follows. All representatives \( g \in \gamma \) of a given left coset \( \gamma \in \Gamma \) carry the given subspace \( \alpha_0 \) into one and the same subspace \( \alpha = g_{\alpha_0} \in A^k \). Writing \( \alpha = \psi(\gamma) \), we have obtained a one-to-one mapping of \( \Gamma \) onto \( A^k \). Composing this mapping \( \psi \) with the canonical mapping \( \varphi \) of \( G \) onto \( \Gamma \), we obtain the following mapping \( \chi = \psi \circ \varphi \) of \( G \) onto \( A^k \):

\[ \chi(g) = g_{\alpha_0} \, . \]

This mapping \( \chi \) is continuous because it is a contraction with respect to the metrics (1) and (5) on \( A^k \) and \( G \):

\[ [\alpha_1, \alpha_2] \leq ||g_1 - g_2|| \, . \]

(6)

In fact, let \( x \in \Omega_{\alpha_1} \), and put \( \xi = g_1^{-1}x \), \( y = g_2 \xi \). Then \( \xi \in \Omega \), \( y \in \Omega_{\alpha_2} \), and

\[ |x - y| = |g_1 \xi - g_2 \xi| \leq ||g_1 - g_2|| \, . \]

Hence \( \rho(\Omega_{\alpha_1}, \Omega_{\alpha_2}) \leq ||g_1 - g_2|| \), and (6) follows by symmetry. The continuity of \( \chi \) implies that of \( \psi \) (cf. N. Bourbaki [2, Chap. I, Théorème 1, p. 53]). Being a one-to-one continuous mapping of the compact space \( \Gamma \) onto \( A^k \), \( \psi \) is a homeomorphism of \( \Gamma \) onto \( A^k \). Moreover, the actions of \( G \) on \( \Gamma \) and on \( A^k \) correspond to each other by the mapping \( \psi \): If \( \gamma_1 = \gamma_2 \), then \( g_{\alpha_1} = \alpha_2 \) (with the notation \( \alpha_1 = \psi(\gamma_1) \), \( \alpha_2 = \psi(\gamma_2) \)). For the purpose of the present note we may, consequently, identify \( \Gamma \) with \( A^k \).

4. Proof of the integral formula (2). By application of the results of § 2 to the homogeneous space \( \Gamma = G/H \) described in § 3 (and identified with \( A^k \)), we obtain

\[ \int_{A^k} F(x) \, d\mu(x) = \int_{\Gamma} F(g_{\alpha_0}) \, d\lambda(g) \]

(7)

for any Baire function \( F \) on \( A^k \) with values \( 0 \leq F(\alpha) \leq +\infty \). Now let \( F \) be defined as in (4) in terms of a given Baire function \( f \) on \( R^n \) (with \( 0 \leq f(x) \leq +\infty \)). Then we have

\[ F(g_{\alpha_0}) = \int_{y \in \alpha_0} f(y) \, d\sigma(y) = \int_{x \in \alpha_0} f(gx) \, d\sigma(x) \, . \]

According to Fubini’s theorem, this represents a Baire function of \( g \in G \), and hence \( F \) is itself a Baire function on \( A^k \) (cf. § 2). Moreover,
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\( \int_G F(gx_0) \, d\lambda(g) = \int_{\alpha_0} \left( \int_G f(gx) \, d\lambda(g) \right) \, d\sigma(x). \)

For \( x \neq 0 \), the inner integral on the right is simply the usual mean value of \( f \) over the sphere \( \Sigma_x \subseteq \mathbb{R}^n \) of centre 0 and passing through the point \( x \). It is, in fact, well known (and may be verified by arguments similar to those employed in \S 3, cf. e.g. C. Chevalley [4, p. 32–33]) that \( \Sigma_x \) may be identified with the homogeneous space \( G/G_x \), where \( G = O(n) \) as before, and \( G_x \) consists of all \( g \in G \) which leave the point \( x \) fixed: \( gx = x \). The invariant integral on \( \Sigma_x \) is, however, the usual mean value, and hence we obtain from (4):

\[ \int_{G} f(gx) \, d\lambda(g) = \frac{\omega}{\omega_n} \int_{\Omega} f(|x| \xi) \, d\omega(\xi), \]

where \( \omega \) is the usual surface measure on the unit sphere \( \Omega \subseteq \mathbb{R}^n \). Inserting this result on the right of (8), and performing the integration over \( \alpha_0 \) by use of polar coordinates (observing that only \(|x| = r \) enters), we obtain in view of (7) the desired identity (2):

\[ \int_{A^k} F(x) \, d\mu(x) = \frac{\omega_k}{\omega_n} \int_0^\infty \int_{\Omega} f(r\xi) \, d\omega(\xi) \int_0^r r^{k-1} \, dr \]

\[ = \frac{\omega_k}{\omega_n} \int_0^\infty \int_{\Omega} f(r\xi) \, \xi^{n-1} \, dr \, d\omega(\xi) \]

\[ = \frac{\omega_k}{\omega_n} \int_{\mathbb{R}^n} |x|^{k-n} f(x) \, dx. \]

5. Applications. Replacing the origin 0 by an arbitrary point \( x \in \mathbb{R}^n \), we obtain for the mean value of the integrals of the function \( f \) over all \( k \)-dimensional linear varieties passing through \( x \) the expression

\[ \frac{\omega_k}{\omega_n} \int_{\mathbb{R}^n} |x - y|^{k-n} f(y) \, dy = \frac{\omega_k}{\omega_n} U^f_k(x), \]

where \( U^f_k \) is the potential of order \( k \) of the "mass distribution" with density \( f \). In the sequel we assume that \((1 + |x|)^{k-n} f(x)\) has a finite integral over \( \mathbb{R}^n \). If the integrals of \( f \) over (almost) all \( k \)-dim. linear varieties in \( \mathbb{R}^n \) are known, then so is, therefore, the potential \( U^f_k \) of \( f \) of order \( k \). For even \( k \), one obtains \( f \) itself (apart from a constant factor) from \( U^f_k \) by applying the Laplace operator \( \Delta \) successively \( k/2 \) times. This follows from the well-known identity due to M. Riesz [9, Chap. I, \S 2]:
\[ \Delta U_k^f = (2-k)(n-k)U_{k-2}^f \quad (2 < k < n), \]

together with Poisson’s formula
\[ \Delta U_2^f = (n-2)\omega_n \cdot f \quad (n > 2), \]
valid in the classical sense provided \( f \) is sufficiently smooth. In the case where \( k \) is odd, one may determine \( U_1^f \) in the above way. Next, \( U_2^f \) is the potential of order 1 of \( U_1^f \) (multiplied by a certain constant, cf. M. Riesz [9, loc. cit.]; we assume here \( n > 2 \)). Finally \( f \) is obtained by application of Poisson’s formula.

Thus the integral formula (2) gives rise to a solution of the problem of determining a function on \( R^n \) when its integrals over all \( k \)-dim. linear varieties in \( R^n \) are given. The first solution of this problem was given by J. Radon [8]. The role of this and related problems in the theory of partial differential equations is described in the book of F. John [7].

A different type of applications of formula (2) arose in the author’s study of “exceptional systems” of surfaces (cf. [5, Chap. 2, § 3]).

REFERENCES


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