IDENTITIES INVOLVING THE PARTITION FUNCTIONS \( q(n) \) AND \( q_0(n) \)

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1. In a previous paper [1] I have, among other results, proved the following three identities involving \( p(n) \), the number of unrestricted partitions of \( n \):

\[
\begin{align*}
(1.1) & \quad 3 \sum_{n=0}^{\infty} p(5n+1)x^n \sum_{n=0}^{\infty} p(5n+2)x^n - 2 \sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+3)x^n \\
& \quad = x \left( \sum_{n=0}^{\infty} p(5n+4)x^n \right)^2,
\end{align*}
\]

\[
(1.2) \quad 2 \sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+2)x^n + x \sum_{n=0}^{\infty} p(5n+3)x^n \sum_{n=0}^{\infty} p(5n+4)x^n \\
& \quad = 2 \left( \sum_{n=0}^{\infty} p(5n+1)x^n \right)^2,
\]

\[
(1.3) \quad \sum_{n=0}^{\infty} p(5n+1)x^n \sum_{n=0}^{\infty} p(5n+3)x^n + \sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+4)x^n \\
& \quad = 2 \left( \sum_{n=0}^{\infty} p(5n+2)x^n \right)^2.
\]

Let \( q(n) \) denote the number of partitions of \( n \) into unequal parts (or, equivalently, the number of partitions into odd parts), and let \( q_0(n) \) denote the number of partitions of \( n \) into odd and unequal parts (which is also the number of self-conjugate partitions). Thus

\[
\begin{align*}
\sum_{n=0}^{\infty} q(n)x^n &= \prod_{n=1}^{\infty} (1 + x^n), \\
\sum_{n=0}^{\infty} q_0(n)x^n &= \prod_{n=1}^{\infty} (1 + x^{2n-1}).
\end{align*}
\]

Tables of \( q(n) \) and \( q_0(n) \) up to \( n = 400 \) have been computed by Watson [2]. Especially, we notice that \( q_0(2) = 0 \).

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The purpose of the present paper is to prove the six identities

\[ (1.4) \quad \sum_{n=0}^{\infty} q(5n) x^n \sum_{n=0}^{\infty} q(5n + 2) x^n = \left( \sum_{n=0}^{\infty} q(5n + 1) x^n \right)^2, \]

\[ (1.5) \quad \sum_{n=0}^{\infty} q(5n) x^n \sum_{n=0}^{\infty} q(5n + 4) x^n = \sum_{n=0}^{\infty} q(5n + 1) x^n \sum_{n=0}^{\infty} q(5n + 3) x^n, \]

\[ (1.6) \quad \sum_{n=0}^{\infty} q(5n + 2) x^n \sum_{n=0}^{\infty} q(5n + 3) x^n = \sum_{n=0}^{\infty} q(5n + 1) x^n \sum_{n=0}^{\infty} q(5n + 4) x^n, \]

\[ (1.7) \quad \sum_{n=0}^{\infty} q_0(5n + 1) x^n \sum_{n=0}^{\infty} q_0(5n + 7) x^n = \left( \sum_{n=0}^{\infty} q_0(5n + 4) x^n \right)^2, \]

\[ (1.8) \quad \sum_{n=0}^{\infty} q_0(5n) x^n \sum_{n=0}^{\infty} q_0(5n + 7) x^n = \sum_{n=0}^{\infty} q_0(5n + 3) x^n \sum_{n=0}^{\infty} q_0(5n + 4) x^n, \]

\[ (1.9) \quad \sum_{n=0}^{\infty} q_0(5n + 1) x^n \sum_{n=0}^{\infty} q_0(5n + 3) x^n = \sum_{n=0}^{\infty} q_0(5n) x^n \sum_{n=0}^{\infty} q_0(5n + 4) x^n. \]

2. We use the notation

\[ \varphi(x) = \prod_{n=1}^{\infty} (1 - x^n). \]

Then we have

\[ (2.1) \quad \sum_{n=0}^{\infty} p(n) x^n = \varphi(x)^{-1}, \]

\[ (2.2) \quad \sum_{n=0}^{\infty} q(n) x^n = \varphi(x^2) \varphi(x)^{-1}, \]

\[ (2.3) \quad \sum_{n=0}^{\infty} q_0(n) x^n = \varphi(-x) \varphi(x^2)^{-1}. \]

Putting

\[ P_s = \sum_{n=0}^{\infty} p(5n + s) x^{5n+s}, \quad s = 0, 1, 2, 3, 4, \]

we get

\[ (2.4) \quad \sum_{n=0}^{\infty} p(n) x^n = P_0 + P_1 + P_2 + P_3 + P_4, \]

where the power series has been divided into five parts, each part consisting of terms whose exponents are congruent (mod 5), the residue class being indicated by the index. The same procedure is used in the equations (2.5)–(2.8), (2.10) and (2.11) below. Let
\[
\sum_{n=0}^{\infty} q(n) x^n = Q_0 + Q_1 + Q_2 + Q_3 + Q_4,
\]

\[
\sum_{n=0}^{\infty} q_0(n) x^n = R_0 + R_1 + R_2 + R_3 + R_4,
\]

\[
\sum_{n=0}^{\infty} p(n) x^{2n} = S_0 + S_1 + S_2 + S_3 + S_4.
\]

From Euler’s identity
\[
\varphi(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{1+n(3n+1)}
\]

it follows that the power series expansion of \(\varphi(x)\) contains no terms with exponents congruent to 3 or 4 \((\text{mod } 5)\). Hence we can write

\[
\varphi(x) = g_0 + g_1 + g_2.
\]

Between these quantities there is a well-known relation, viz. (see [3, p. 102] and [1, p. 84])

\[
g_0g_2 = -g_1^2.
\]

Further, we put

\[
\varphi(x^2) = a_0 + a_2 + a_4,
\]

\[
\varphi(-x) = b_0 + b_1 + b_2.
\]

We also need the identities

\[
\varphi(x^2)\varphi(x)^{-1} = \sum_{n=0}^{\infty} x^{1+n(n+1)},
\]

\[
\varphi(-x^2)\varphi(x^2)^{-1} = \sum_{n=-\infty}^{\infty} x^{2n^2},
\]

which follow from Jacobi’s formula
\[
\Pi_{n=1}^{\infty} (1-z^{2n})(1+yz^{2n-1})(1+y^{-1}z^{2n-1}) = \sum_{n=-\infty}^{\infty} y^n z^{n^2}.
\]

3. In this section we shall prove the identities (1.4)–(1.6). From (2.2) we get, using (2.1), (2.4), (2.5) and (2.10),

\[
Q_0 + Q_1 + Q_2 + Q_3 + Q_4 = (a_0 + a_2 + a_4)(P_0 + P_1 + P_2 + P_3 + P_4),
\]

and hence
\[ Q_0 = a_0 P_0 + a_2 P_3 + a_4 P_1, \]
\[ Q_1 = a_0 P_1 + a_2 P_4 + a_4 P_2, \]
\[ Q_2 = a_0 P_2 + a_2 P_0 + a_4 P_3, \]
\[ Q_3 = a_0 P_3 + a_2 P_1 + a_4 P_4, \]
\[ Q_4 = a_0 P_4 + a_2 P_2 + a_4 P_0. \]

Further, by (2.2), (2.5), (2.10) and (2.12)
\[
(a_0 + a_2 + a_4)(Q_0 + Q_1 + Q_2 + Q_3 + Q_4) = \sum_{n=0}^{\infty} x^{4n(n+1)}. 
\]

This power series contains no terms with exponents congruent to 2 or 4 (mod 5), and therefore we have
\[ a_0 Q_2 + a_2 Q_0 + a_4 Q_3 = 0, \]
\[ a_0 Q_4 + a_2 Q_2 + a_4 Q_0 = 0. \]

The identities (1.1)–(1.3) can now be written
\[ 3P_1 P_2 - 2P_0 P_3 - P_4^2 = 0, \]
\[ P_0 P_2 + P_3 P_4 - 2P_1^2 = 0, \]
\[ P_1 P_3 + P_0 P_4 - 2P_2^2 = 0. \]

Finally, by (2.8)–(2.10) we get
\[ a_0 a_4 = -a_2^2. \]

Now the identities (1.4)–(1.6) can be deduced from the system (3.1)–(3.11) by elimination of the quantities \( a_s \) and \( P_s \). We proceed as follows: From (3.1)–(3.3) we find, using (3.11)
\[ Q_0 Q_2 - Q_1^2 = a_0^2 (P_0 P_2 - P_1^2) + a_2^2 (P_1 P_2 - P_3^2) + a_4^2 (P_1 P_3 - P_4^2) + a_0 a_2 (P_0^2 + P_2 P_3 - 2P_1 P_4) + a_2 a_4 (P_3^2 + P_0 P_1 - 2P_2 P_4). \]

Inserting the expressions (3.1) and (3.3)–(3.5), we get from (3.6) and (3.7)
\[ a_0^2 P_2 + 2a_0 a_2 P_0 - a_2^2 P_3 = -a_4^2 P_4 - 2a_2 a_4 P_1, \]
\[ a_4^2 P_1 + 2a_2 a_4 P_3 - a_2^2 P_0 = -a_0^2 P_4 - 2a_0 a_2 P_2. \]

Multiplication of (3.13) and (3.14) yields
\[ 0 = -a_0^2 P_0 P_2 + a_2^2 (5P_1 P_2 - 3P_0 P_3 - P_4^2) - a_4^2 P_1 P_3 + 2a_0 a_2 (P_1 P_4 - P_2 P_3 - P_0^2) + 2a_2 a_4 (P_2 P_4 - P_0 P_1 - P_3^2). \]
Further we multiply (3.13) and (3.14) by $P_0$ and $P_3$ respectively, and add. Thus we obtain

$$0 = a_0^2(P_0P_2 + P_3P_4) - 2a_2^2P_0P_3 + a_4^2(P_1P_3 + P_0P_4) + 2a_0a_2(P_0^2 + P_2P_3) + 2a_2a_4(P_3^2 + P_0P_1),$$

which by (3.9) and (3.10) reduces to

$$(3.16) \quad 0 = a_0^2P_1^2 - a_2^2P_0P_3 + a_4^2P_2^2 + a_0a_2(P_0^2 + P_2P_3) + a_2a_4(P_3^2 + P_0P_1).$$

Adding the equations (3.12), (3.15) and (3.16), we get

$$Q_0Q_2 - Q_1^2 = 2a_2^2(3P_1P_2 - 2P_0P_3 - P_4^2).$$

Hence, by (3.8)

$$(3.17) \quad Q_0Q_2 = Q_1^2,$$

which proves (1.4).

From (3.1), (3.2), (3.4) and (3.5) we find

$$(3.18) \quad Q_0Q_4 - Q_1Q_3 = a_0^2(P_0P_4 - P_1P_3) + a_2^2(2P_2P_3 - P_1P_4 - P_0^2) + a_4^2(P_0P_1 - P_2P_4) + a_0a_2(P_0P_2 - P_1^2) + a_2a_4(P_3P_0 - P_4^2).$$

We multiply the equations (3.13) and (3.14) by $P_2$ and $P_0$ respectively and subtract. Thus we obtain

$$(3.19) \quad 0 = a_0^2(P_2^2 - P_0P_4) + a_2^2(P_0^2 - P_2P_3) + a_4^2(P_2P_4 - P_0P_1) + 2a_2a_4(P_1P_2 - P_0P_3).$$

Multiplying (3.19) by 2 and adding it to (3.18) we get, using (3.8) and (3.10)

$$(3.20) \quad Q_0Q_4 - Q_1Q_3 = a_2^2(P_0^2 - P_1P_4) + a_4^2(P_2P_4 - P_0P_1) + a_0a_2(P_0P_2 - P_1^2) + a_2a_4(P_3P_0 - P_4^2).$$

Similarly we find

$$(3.21) \quad Q_2Q_3 - Q_1Q_4 = a_0^2(P_1P_4 - P_2P_3) + a_2^2(P_3^2 - P_2P_4) + a_0a_2(P_1P_2 - P_0P_3) + a_2a_4(P_1P_3 - P_2^2).$$

From (3.20) and (3.21) follows

$$\begin{align*}
&\quad a_0a_2^{-1}(Q_0Q_4 - Q_1Q_3) + a_4a_2^{-1}(Q_2Q_3 - Q_1Q_4) \\
&= a_0^2(P_0P_2 - P_1^2) + 2a_2^2(P_0P_3 - P_1P_2) + a_4^2(P_1P_3 - P_2^2) + a_0a_2(P_0^2 + P_2P_3 - 2P_1P_4) + a_2a_4(P_3^2 + P_0P_1 - 2P_2P_4) \\
&= Q_0Q_2 - Q_1^2,
\end{align*}$$
by (3.12) and (3.8). Hence

(3.22) \[ a_0(Q_0 Q_4 - Q_1 Q_3) + a_4(Q_2 Q_3 - Q_1 Q_4) = 0. \]

If we multiply this equation by \( Q_0 \) and replace \( Q_0 Q_2 \) by \( Q_1^2 \), we get

\[ (a_0 Q_0 - a_4 Q_1)(Q_0 Q_4 - Q_1 Q_3) = 0. \]

Obviously, \( a_0 Q_0 - a_4 Q_1 \neq 0 \). Hence

(3.23) \[ Q_0 Q_4 = Q_1 Q_3, \]

and consequently, by (3.22) (or by (3.17))

(3.24) \[ Q_2 Q_3 = Q_1 Q_4. \]

Thus the identities (1.5) and (1.6) are proved.

4. It remains to prove (1.7)--(1.9). From (2.3) we get, using (2.6), (2.7) and (2.11)

\[ R_0 + R_1 + R_2 + R_3 + R_4 = (b_0 + b_1 + b_2)(S_0 + S_1 + S_2 + S_3 + S_4), \]

and hence

(4.1) \[ R_0 = b_0 S_0 + b_1 S_4 + b_2 S_3, \]
(4.2) \[ R_1 = b_0 S_1 + b_1 S_0 + b_2 S_4, \]
(4.3) \[ R_2 = b_0 S_2 + b_1 S_1 + b_2 S_0, \]
(4.4) \[ R_3 = b_0 S_3 + b_1 S_2 + b_2 S_1, \]
(4.5) \[ R_4 = b_0 S_4 + b_1 S_3 + b_2 S_2. \]

Further, by (2.3), (2.6), (2.11) and (2.13)

\[ (b_0 + b_1 + b_2)(R_0 + R_1 + R_2 + R_3 + R_4) = \sum_{n=0}^{\infty} x^n. \]

From this we conclude

(4.6) \[ b_0 R_2 + b_1 R_1 + b_2 R_0 = 0, \]
(4.7) \[ b_0 R_3 + b_1 R_2 + b_2 R_1 = 0. \]

Replacing \( x \) by \( x^2 \) in (3.8)--(3.10), we obtain

(4.8) \[ 3S_2 S_4 - 2S_0 S_1 - S_3^2 = 0, \]
(4.9) \[ S_0 S_4 + S_1 S_3 - 2S_2^2 = 0, \]
(4.10) \[ S_1 S_2 + S_0 S_3 - 2S_4^2 = 0. \]

Finally, by (2.8), (2.9) and (2.11)
\[(4.11) \quad b_0 b_2 = -b_1^2.\]

It is easily seen that the system (3.1)–(3.11) is changed into the system (4.1)–(4.11) by the substitutions

\[
\begin{array}{ccc}
P_0 & \rightarrow & S_0 \\
P_1 & \rightarrow & S_2 \\
P_2 & \rightarrow & S_4 \\
P_3 & \rightarrow & S_1 \\
P_4 & \rightarrow & S_3 \\
Q_0 & \rightarrow & R_2 \\
Q_1 & \rightarrow & R_4 \\
Q_2 & \rightarrow & R_1 \\
Q_3 & \rightarrow & R_3 \\
Q_4 & \rightarrow & R_0 \\
a_0 & \rightarrow & b_2 \\
a_1 & \rightarrow & b_1 \\
a_2 & \rightarrow & b_1 \\
a_4 & \rightarrow & b_0 \\
\end{array}
\]

Now, the equations (3.17), (3.23) and (3.24) were deduced from (3.1)–(3.11), and from these alone. Hence, the system (4.1)–(4.11) implies the equations

\[
\begin{align*}
R_1 R_2 &= R_4^2, \\
R_0 R_2 &= R_3 R_4, \\
R_1 R_3 &= R_0 R_4,
\end{align*}
\]

and thus the identities (1.7)–(1.9) are proved.

REFERENCES


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