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ON A GENERAL THEORY OF INTEGRATION BASED ON ORDER

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Introduction.

Integration theory is usually confined to linear lattices of real valued functions, and the theory is based on the linear structure as well as the order structure and the topological structure of simple (pointwise) convergence. In the present paper we shall verify that the fundamental results of integration theory may be stated and proved by means of order properties only. Linearity or topological concepts will not be applied.

In § 1 we shall define the concept of an integral over an (abstract) lattice. It is defined to be a lattice-valuation with an additional limiting property (axiom I_3 of § 1). This concept generalises the elementary integral of M. H. Stone [8]. We shall then define a full integral over a lattice to be a lattice-integral for which the Beppo Levi theorem is valid (I_4 of § 1). Then we shall prove that a full integral over a lattice has all the fundamental limiting properties of the Lebesgue integral, i.e. the properties stated in the Fatou, Lebesgue, and Riesz-Fischer theorems.

In § 2 we shall introduce the notions of upper and lower semi-integrals (axioms UI_1 to UI_3 of § 2). These concepts generalise the upper and lower integrals as well as the outer and inner measures in various existing approaches to integration theory (ref. [3] to [8]). Prop. 3 of § 2 states that under fairly general conditions a pair consisting of an upper and a lower semi-integral will determine a full integral.

In § 3 we shall apply the results of § 2 to prove that every integral defined over a lattice imbedded in a countably distributive σ-lattice (3.1), can be extended to a full integral. The paragraphs 3 and 4 outline generalisations of the Stone–Daniell theory and the Bourbaki theory, respectively.

In § 5 we shall prove that under fairly general conditions, there exists a unique, minimal, full extension of a given integral, and that the extensions of § 3 and § 4 are only unessentially different from this minimal

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extension. (Cf. theorem 7 of § 5 for the precise content of the last statement.)

§ 1. General integrals.

Let L denote a lattice with elements $x, y, z, \ldots$. To specify an increasing sequence of elements of L we shall use the notation $\{y_n\}\uparrow$. If $\{y_n\}\uparrow$ has a l.u.b. y, then we shall write $y_n \uparrow y$. The dual concepts are defined analogously.

We shall write $(\{y_n\}\uparrow) \geqq (\{z_n\}\downarrow)$ if $y \geqq y_n$, $z \leqq z_n$ for all natural n, implies $y \geqq z$. If $y_n \uparrow y$ and $z_n \downarrow z$, then $(\{y_n\}\uparrow) \geqq (\{z_n\}\downarrow)$ means $y \geqq z$.

An *integral* I over L is a real valued function over L with the following properties:

(I_1) $x \geqq y \Rightarrow I(x) \geqq I(y)$,
(I_2) $I(x)+I(y)=I(x \vee y)+I(x \wedge y)$,
(I_3) $(\{y_n\}\uparrow) \geqq (\{z_n\}\downarrow) \Rightarrow \sup_n I(y_n) \geqq \inf_n I(z_n)$.

(I_1) and (I_2) together define a *valuation* on L (cf. [1, p. 74]). For a general valuation v, the function

$$d(x, y) = v(x \vee y) - v(x \wedge y) \tag{1.1}$$

is a pseudo-metric. This result is established by means of the relation (1.2) which is easily proved from (I_1) and (I_2) and remains valid if v is allowed to assume one of the values $+\infty$, $-\infty$ (cf. [1, p. 76).

$$d(x \vee z,\ y \wedge z) + d(x \vee z,\ y \wedge z) \leqq d(x, y)\ . \tag{1.2}$$

It is easily verified that the function v is continuous with respect to (1.1).

A consequence of (I_1) and (I_3) is the following statement, known as Daniell's axiom:

$$x_n \uparrow x \Rightarrow I(x_n) \uparrow I(x), \qquad \text{and dually .} \tag{1.3}$$

Proposition 1. If L is a linear lattice of real valued functions defined over some set S, and I is a positive linear functional over L satisfying (1.3), then I is an integral over L.

Proof. (I_2) is a consequence of the identity $x+y=x \vee y + x \wedge y$ generally valid in lattice-ordered groups [1]. If $(\{y_n\}\uparrow) \geqq (\{z_n\}\downarrow)$, then for every $t \in S$ we have $\sup_n y_n(t) \geqq \inf_n z_n(t)$, and hence $\sup_n [y_n(t) - z_n(t)] \geqq 0$, which implies $(y_n - z_n) \wedge 0 \uparrow 0$. Thus by (1.3)

$$\sup_n I(y_n - z_n) \geqq \sup_n I[(y_n - z_n) \wedge 0] = 0\ ,$$

and hence $\sup_n I(y_n) \geqq \inf_n I(z_n)$, q.e.d.

An integral I over a lattice L is said to be *full* (over L) if

(I_4) $\{x_n\}\uparrow$, $\sup_n I(x_n)<\infty \Rightarrow x_n\uparrow x\in L$, and dually.

(The property required in I_4 may be called the "Beppo Levi property".)

PROPOSITION 2. A real valued function I over L which satisfies (I_1), (I_2), (1.3) and (I_4) is a full integral.

PROOF. Let $(\{y_n\}\uparrow)\geqq(\{z_n\}\downarrow)$. If $\sup_n I(y_n)=+\infty$ or $\inf_n I(z_n)=-\infty$, then the inequality at the end of (I_3) is trivially satisfied. If not, we have by (I_4), (I_1) and (1.3): $y_n\uparrow y$, $z_n\downarrow z$, and $\sup_n I(y_n)=I(y)\geqq I(z)=\inf_n I(z_n)$, q.e.d.

LEMMA. If I is a full integral over L, $x_0\leqq x_n$ for all natural n, and $\liminf_n I(x_n)=\alpha<\infty$; then $y_n=\inf_{m\geqq n}x_n\in L$, $y_n\uparrow x=\liminf_n x_n\in L$, and $I(y_n)\uparrow I(x)\leqq\alpha$.

In particular, the conditions stated are satisfied if $\{x_n\}$ is some sequence such that $d(x_n, x_{n+1})\leqq 2^{-n}$ for all natural n, and in this case $d(x_n, x)\to 0$.

PROOF. 1) We write $y_{n,r}=x_n\wedge x_{n+1}\wedge\ldots\wedge x_{n+r}$ for all natural n, r. Then $\{y_{n,r}\}\downarrow$, and $\inf_r I(y_{n,r})\geqq I(x_0)$ for every fixed n. Hence by (I_4) we have $y_{n,r}\downarrow y_n\in L$.

For every fixed n, $y_n\leqq x_m$ for $m\geqq n$. Hence (by (I_1)) $I(y_n)\leqq\inf_{m\geqq n}I(x_m)$. Since $\sup_n\inf_{m\geqq n}I(x_m)=\alpha<\infty$, we can apply (I_4) to the sequence $\{y_n\}\uparrow$, giving $y_n\uparrow x\in L$. Finally, application of (1.3) to the same sequence gives $I(x)=\sup_n I(y_n)\leqq\alpha$.

2) Application of (1.2) gives

$$(1.4)\quad \begin{aligned} d(x_n, y_{n,r}) &\leqq \sum_{i=0}^{r-1} d(y_{n,i}, y_{n,i+1}) \\ &= \sum_{i=0}^{r-1} d(y_{n,i}\wedge x_{n+i}, y_{n,i}\wedge x_{n+i+1}) \leqq \sum_{i=n}^{n+r-1} d(x_i, x_{i+1}) . \end{aligned}$$

Hence by hypothesis $d(x_n, y_{n,r})\leqq 2^{-n+1}$ for all natural r. Since $x_n\geqq y_{n,r}$, this means $I(x_n)-I(y_{n,r})\leqq 2^{-n+1}$, or equivalently $I(y_{n,r})\geqq I(x_n)-2^{-n+r}$ for all r. Hence by (I_4) $y_{n,r}\downarrow y_n\in L$, and by (1.3) $I(y_{n,r})\downarrow I(y_n)$. Thus

$$(1.5)\qquad d(x_n, y_n) = I(x_n)-I(y_n) = I(x_n)-\inf_r(y_{n,r}) \leqq 2^{-n+1} .$$

In particular we may take the lower bound x_0 to be y_1. Further

$$|I(x_n)-I(x_{n+1})| \leqq d(x_n, x_{n+1}) \leqq 2^{-n} ,$$

proving $\{I(x_n)\}$ to be a Cauchy sequence, and thus $\liminf_n I(x_n)<\infty$.

By application of the first half of the lemma and (1.5), we finally obtain

$$d(x_n, x) \leqq d(x_n, y_n) + d(y_n, x) \to 0, \text{ q.e.d.}$$

THEOREM 1 (Fatou). *A valuation v over L is a full integral if and only if*

$$(1.6) \quad x_0 \leqq x_n,\ n = 1, 2, \ldots,\ \liminf_n v(x_n) = \alpha < \infty \Rightarrow \liminf_n x_n = x \in L,\ v(x) \leqq \alpha,$$

and dually.

PROOF. 1) If $\{x_n\}\uparrow$ and $\sup_n v(x_n) < \infty$, then application of (1.6) gives $x_n \uparrow x$ and $v(x) \leqq \sup_n v(x_n)$, proving (I_4) and (1.3), so that v is a full integral (cf. proposition 2).

2) If v is a full integral, then (1.6) is valid in virtue of the lemma.

THEOREM 2 (Lebesgue). *If I is a full integral over L, L is imbedded in a lattice H, $y \in L$, $z \in L$, $x_n \in L$, $y \leqq x_n \leqq z$ for $n = 1, 2, \ldots$, and* $\liminf_n x_n = \limsup_n x_n = x \in H$; *then we can conclude that x also belongs to L and* $\lim_{n\to\infty} I(x_n) = I(x)$.

PROOF. Application of (1.6).

THEOREM 3 (Riesz-Fischer). *If I is a full integral over L, then L is complete with respect to the pseudo-metric defined by I* (1.1).

PROOF. Every Cauchy sequence contains a subsequence for which $d(x_n, x_{n+1}) \leqq 2^{-n}$ for every n. Application of the lemma accomplishes the proof.

§ 2. Generation of full integrals by means of upper and lower semi-integrals.

An *upper semi-integral* I^* over L is an extended (that is to $\pm\infty$) real valued function over L, with the properties:

(UI_1) $x \leqq y \Rightarrow I^*(x) \leqq I^*(y)$.

(UI_2) $I^*(x) + I^*(y) \geqq I^*(x \vee y) + I^*(x \wedge y)$. (Both sides simultaneously undetermined.)

(UI_3) $x_n \uparrow x \Rightarrow I^*(I_n) \uparrow I^*(x)$. (Not dually.)

A *lower semi-integral* is defined by reversing the sign of inequality in (UI_2) and the arrows in (UI_3).

PROPOSITION 3. If I^* and I_* are upper and lower semi-integrals defined on a σ-lattice H, and $I_*(x) \leqq I^*(x)$ for all x, then

$$\tilde{L} = \{x \mid x \in H, \ -\infty < I_*(x) = I^*(x) < +\infty\}$$

is either empty or a sublattice of H to which the common restriction $\tilde{I}$ of I^* and I_* is a *full* integral.

PROOF. 1) If $x \in \tilde{L}$ and $y \in \tilde{L}$, then

$$\tilde{I}(x)+\tilde{I}(y) \leqq I_*(x \vee y)+I_*(x \wedge y) \leqq I^*(x \vee y)+I^*(x \wedge y) \leqq \tilde{I}(x)+\tilde{I}(y) .$$

Thus, the signs of equality are valid, which is possible only if $I^*(x \vee y) = I^*(x \vee y)$, $I_*(x \wedge y) = I^*(x \wedge y)$, and these values both are finite. Hence if $\tilde{L} \neq \emptyset$, then $\tilde{L}$ is a sublattice of H.

2) $\tilde{L}$ evidently satisfies (I_1) and (I_2). By prop. 2 of § 1, it is sufficient to prove (1.3) and (I_4) to make sure that I is a full integral. Let $\{x_n\}\uparrow$, $x_n \in \tilde{L}$ for $n=1, 2, \ldots$. Since H is σ-complete $x_n \uparrow x \in H$. Then by (UI_1) for I_* and (UI_3) for I^*

$$\sup_n \tilde{I}(x_n) \leqq I_*(x) \leqq I^*(x) = \sup_n \tilde{I}(x_n) .$$

Thus the signs of equality are valid proving (1.3) and (I_4).

Proposition 3 gives a clue to the extension problem. As we shall see in the subsequent paragraphs, integrals can be extended to upper and lower semi-integrals in very general cases by means of various limiting processes. The major difficulty remains to extend them as far as to σ-lattices so that proposition 3 can be applied.

LEMMA. If H is a σ-lattice, $U^\circ \subset H$, U° is closed with respect to finite meets and countable joins, and I° is an upper integral over U° which even satisfy (I_2) and not only (UI_2), and takes values in $]-\infty, +\infty]$; then I° can be extended to an upper integral I^* defined over H by the expression

$$I^*(x) = \begin{cases} \inf I^\circ(y), \ x \leqq y \in U^\circ, \text{ if such a } y \text{ exists}, \\ +\infty, \text{ else}. \end{cases} \tag{2.1}$$

PROOF. I^* evidently satisfies (UI_1). To prove (UI_2), we consider $x \in H$, $y \in H$, and we first suppose $x \leqq x' \in U^\circ$, $y \leqq y' \in U^\circ$. Then $x \vee y \leqq x' \vee y'$, $x \wedge y \leqq x' \wedge y'$. Hence

$$I^*(x \vee y)+I^*(x \wedge y) \leqq I^\circ(x' \vee y')+I^\circ(x' \wedge y') = I^\circ(x')+I^\circ(y') .$$

Passing to the minimal value at the right, we obtain the desired statement.

To prove (UI_2) when no such x', y' exist, we have to examine a few simple cases.

We then turn to (UI_3). Let $x_n \uparrow x$. Then the non-trivial part of the verification consists in proving $I^*(x) \leqq \sup_n I^*(x_n)$. If $\sup_n I^*(x_n) = +\infty$, this statement is trivial. If not, we may assume (extracting a subsequence if necessary) that $I^*(x_{n+1}) - I^*(x_n) \leqq 2^{-n}$ for every n.

Now for $n = 1, 2, \ldots$ we define x_n' such that

$$x_n \leqq x_n' \in U^\circ, \quad I^\circ(x_n') - I^*(x_n) \leqq 2^{-n} .$$

From the assumptions concerning I°, we know $d^\circ(x, y) = I^\circ(x \vee y) - I^\circ(x \wedge y)$ to be a pseudo-metric on U° (cf. § 1). We then have

$$\begin{aligned}
d^\circ(x_n', x_{n+1}') &= I^\circ(x_n') + I^\circ(x_{n+1}') - 2I^\circ(x_n' \wedge x_{n+1}') \\
&\leqq I^\circ(x_n') + I^\circ(x_{n+1}') - 2I^*(x_n) \\
&= [I^\circ(x_n') - I^*(x_n)] + [I^\circ(x_{n+1}') - I^*(x_{n+1})] + [I^*(x_{n+1}) - I^*(x_n)] \\
&\leqq 3 \cdot 2^{-n} .
\end{aligned}$$

We now write $y_n = \sup_{m \geqq n} x_m' \in U^\circ$, and an expansion similar (actually dual) to (1.4) gives $d^\circ(x_n', y_n) \leqq 3 \cdot 2^{-n+1}$ for all n. Since $x_n' \leqq y_n$, this means

$$I^\circ(y_n) - I^\circ(x_n') \leqq 3 \cdot 2^{-n+1} .$$

Moreover $x \leqq y_n$ for all n, and hence

$$I^*(x) \leqq I^\circ(y_n) \leqq I^\circ(x_n') + 3 \cdot 2^{-n+1} \leqq I^*(x_n) + 7 \cdot 2^{-n} .$$

Thus, we have obtained $I^*(x) \leqq \lim_{n \to \infty} I^*(x_n)$, q.e.d.

§ 3. First application. Extension by sequences. Generalised Stone-Daniell theory.

In this paragraph we shall assume I to be an integral defined over a lattice L which is imbedded in a σ-lattice H satisfying the following requirement of *countable distributivity*:

$$(3.1) \qquad (\sup_n x_n) \wedge (\sup_m y_m) = \sup_{m,n} (x_n \wedge y_m), \quad \text{and dually} .$$

THEOREM 4. *The set of those $x \in H$ for which*

$$(3.2) \qquad \begin{aligned} -\infty < \sup\{\inf_n I(z_n) \mid z_n \in L,\ z_n \downarrow z \leqq x\} \\ = \inf\{\sup_n (y_n) \mid y_n \in L,\ y_n \uparrow y \geqq x\} < +\infty \end{aligned}$$

is a sublattice $\tilde{L}$ of H, over which the common value of the two expressions of (3.2) *defines a full integral $\tilde{I}$ extending I. Moreover L is dense in $\tilde{L}$ with respect to the pseudo-metric $\tilde{d}(x, y) = \tilde{I}(x \vee y) - \tilde{I}(x \wedge y)$.*

PROOF. We define $U^\circ \subset H$ as the set of those x which are of the form $x = \sup_n y_n$, $y_n \in L$, $n = 1, 2, \ldots$. The set U° is closed with respect to finite meets and countable joins. A function I° is defined on U° by

$$(3.3)\quad I^\circ(x) = \sup\{I(y_n) \mid y_n \uparrow x,\ y_n \in L\} = \sup\{I(y) \mid x \geqq y \in L\}\,.$$

Application of (1.3) proves I° to be an extension of I, that is, $I^\circ(x) = I(x)$ for $x \in L$.

We shall verify that I° satisfies the requirements of the lemma in § 2. Evidently (UI_1) is satisfied. To prove (I_2), we suppose $x \in U^\circ$, $y \in U^\circ$, $x_n \uparrow x$, $y_n \uparrow y$, $x_n \in L$, $y_n \in L$, $n = 1, 2, \ldots$. Then application of (I_2) to x_n, y_n and passage to the limit as $n \to \infty$ gives the desired result.

To prove (UI_3), we suppose $x_n \in U^\circ$, $x_n \uparrow x$. Let $y_{n,i} \uparrow x_n$, $y_{n,i} \in L$, and let us write $z_i = \sup_{n,j \leqq i} y_{n,j}$. Then

$$z_i \in L, \quad z_i \leqq x_i \qquad \text{for} \qquad i = 1, 2, \ldots,$$

and $z_i \uparrow x$. Thus $I(z_i) \leqq I^\circ(x_i)$, and further $\sup_i I(z_i) \leqq \sup_i I^\circ(x_i)$. Since $z_i \uparrow x$, we have

$$I^\circ(x) = \sup_i I(z_i) \leqq \sup_i I^\circ(x_i)\,.$$

The converse inequality, $\sup_i I^\circ(x_i) \leqq I^\circ(x)$, is a consequence of (UI_1), and (UI_3) is proved.

We can now apply the lemma of § 2 to define an upper integral over H. Substitution of the actual value of I° into (2.1) shows that the value $I^*(x)$ for some $x \in H$ can be written like the latter of the two expressions of (3.2). Since I° is an extension of I, I^* must be an extension of I as well.

Similarly we can verify that the former of the two expressions of (3.2) defines a lower integral I_*, which is also an extension of I.

By application of the axiom (I_3) we obtain $I_*(x) \leqq I^*(x)$. (Now, for the first time, we really need (I_3) and not only (1.3).)

From prop. 3 of § 2 we can conclude that $\tilde{L} \supset L$ is a sublattice of H, and that the common restriction $\tilde{I}$ of I^* and I_* to L is a full integral.

It remains to be proved that L is dense in $\tilde{L}$. Let $x \in \tilde{L}$ and $\varepsilon > 0$. Let $x' \in U^\circ$, $x' \geqq x$, $I^\circ(x') - I^*(x) < \frac{1}{2}\varepsilon$. Let also $y \in L$, $y \leqq x'$, $I^\circ(x') - I(y) < \frac{1}{2}\varepsilon$. Then $\tilde{d}(x, y) = \tilde{d}(x, x') + \tilde{d}(x', y) < \varepsilon$.

§ 4. Second application. Extension by directed sets. Generalized Bourbaki theory.

A subset Φ of a lattice L (or more generally, of a partially ordered set) is said to be *directed upward* if any pair x, y of elements of Φ admits a common majorant $z \in \Phi$. The notion *directed downward* is defined dually.

The notion of 'directed set' is effectively, though not formally, a generalisation of the notion of monotoneous sequence. The incongruity could easily be removed by replacing Φ by a family $\{x_\alpha\}_{\alpha \in A}$, where A is some partially ordered index-set, and $\alpha \to x_\alpha$ is an order-isomorphism of A onto Φ. However, such a convention would only complicate our notations, and shall not be adopted.

To specify a set Φ which is directed upward, we shall use the notation $\Phi \uparrow$, and dually $\Psi \downarrow$. If $\sup_{x \in \Phi} x = y$ exists in L, we shall write $\Phi \uparrow y$, and dually $\Psi \downarrow z$. If $\Phi \uparrow$, $\Psi \downarrow$ and $y \geqq \Phi$, $z \leqq \Psi$ implies $y \geqq z$, then we write $(\Phi \uparrow) \geqq (\Psi \downarrow)$.

In this section we shall consider integrals satisfying the following strengthened version of (I_3)

$$(I_3') \qquad (\Phi \uparrow) \geqq (\Psi \downarrow) \Rightarrow \sup_{x \in \Phi} I(x) \geqq \inf_{x \in \Psi} I(x) .$$

A consequence of (I_1) and (I_3') is

$$(4.1) \qquad \Phi \uparrow y \in L \Rightarrow \sup_{x \in \Phi} I(x) = I(y) ,$$

and dually.

(1.3) and (4.1) are the axioms called MA' and MA ("mesure abstraite") in [3, p. 114].

PROPOSITION 4. If L is a linear lattice of real valued functions defined over some set S, and I is a positive, linear functional over L satisfying (4.1), then I is an integral over L satisfying (I_3').

PROOF. Minor modifications in the proof of prop. 1.

There exist important examples of integrals not satisfying (I_3') (cf. f. ex. [7]). The most important integrals which do satisfy (I_3'), are the positive, linear functionals over the linear lattice of continuous real valued functions with compact support on some locally compact set (cf. [3, p. 54 and p. 105] [7] [8]).

In the remaining part of this section we shall assume I to be an integral which satisfes (I_3') and is defined on a lattice L imbedded in a complete lattice H satisfying the following requirement of infinite distributivity:

$$(4.2) \qquad (\sup_{\alpha \in A} x_\alpha) \wedge (\sup_{\beta \in B} y_\beta) = \sup_{\alpha \in A,\ \beta \in B} (x_\alpha \wedge y_\beta), \qquad \text{and dually} .$$

THEOREM 5. *The set of those $x \in H$ for which*

$$(4.3)\qquad -\infty < \sup\left\{\inf_{z\in\Psi} I(z) \mid \Psi \subset L,\ \Psi\downarrow z_0 \leqq x\right\}$$
$$= \inf\left\{\sup_{y\in\Phi} I(y) \mid \Phi \subset L,\ \Phi\uparrow y_0 \geqq x\right\} < +\infty$$

is a sublattice $\bar{L}$ *of* H, *over which the common value of the two expressions of* (4.3) *defines a full integral* $\bar{I}$ *extending* I. *Moreover* L *is dense in* $\bar{L}$ *with respect to the pseudo-metric* $\bar{d}(x, y) = \bar{I}(x \vee y) - \bar{I}(x \wedge x)$.

The PROOF is a slightly modified version of the proof of theorem 4. The set U° is replaced by V° consisting of all $x \in H$ for which $x = \sup_{y\in A} y$, $A \subset L$. The function I° is replaced by J° defined on V° by:

$$(4.4)\quad J^\circ(x) = \sup\{I(y) \mid y \in \Phi \subset L,\ \Phi\uparrow y\} = \sup\{I(y) \mid x \geqq y \in L\}.$$

Except for the verification that J° satisfies (UI_3), which we shall give in complete detail, the remaining proof only differs unessentially from the previous one.

Let $x_n \in V^\circ$, $x_n \uparrow x$. Let $\Phi_n \uparrow x_n$, $\Phi_n \subset L$, and let us write Φ_n' for the directed set of all finite joins from $\bigcup_{i=1}^{n} \Phi_i$, and Φ' for the directed sets of all finite joins from $\bigcup_{i=1}^{\infty} \Phi_i$. Then $\Phi_n' \subset L$, $\Phi_n' \uparrow x_n$, and $\Phi' \subset L$, $\Phi' \uparrow x$. Thus for every n, $\sup_{z\in\Phi_n'} I(z) = J^\circ(x_n)$. Passing to the supremum-values as n varies, we obtain $\sup_{z\in\Phi'} I(z) = \sup_n J^\circ(x_n)$. Since $\Phi' \uparrow x$, this means $J^\circ(x) = \sup_n J^\circ(x_n)$, q.e.d.

REMARKS. The extended integral $\bar{I}$ does not generally satisfy (I_3'). This is not even so in the classical case where I is the Riemann integral over the class of continuous real functions with compact support, and $\bar{I}$ becomes the Lebesgue integral (like $\tilde{I}$ of § 3 as well) (cf. [7]).

It follows from (3.2) and (4.3) that $\tilde{L} \subset \bar{L}$ and that $\tilde{I}(x) = \bar{I}(x)$ for $x \in \tilde{L}$. In the special case where I is a positive linear functional over the class of continuous real valued functions over some compact set S, $\tilde{I}$ may be defined by a "Baire measure", and $\bar{I}$ by a "Borel measure" (Halmos' terminology) (cf. [5, p. 223] [7] [8]). In this particular case the set V° of the above proof may be interpreted as the class of all lower semi-continuous functions. By means of this interpretation it can be proved that whenever S satisfies the second axiom of countability the sets U° and V°, and hence also $\tilde{L}$ and $\bar{L}$ will coalesce (cf. [2, p. 12]).

§ 5. Uniqueness.

All lattices occuring in the present section are supposed to be contained in a fixed lattice H which is (at least) σ-complete and countably distributive. The only lattice operations which will be considered, are the restrictions of those on H. Accordingly we shall use the word *lattice* to mean a subset of H closed with respect to the restrictions of the finite lattice-operations on H, and the word *σ-lattice* to mean a subset closed with respect to the restrictions of the enumerable lattice operations on H. These conventions conform with the interpretation of H as the set of all extended real valued functions over some set, and the ordinary use of the symbols $f \vee g$, $f \wedge g$ for extended real valued functions.

The symbol $C(L)$ shall denote the smallest σ-lattice containing the lattice L. With the functional interpretation, $C(L)$ becomes the class of Baire functions determined by L (cf. [6, p. 32]).

LEMMA 1. If $x \in C(L)$, then there exists a sequence $y_n \in L$, $y_n \leqq y_{n+1}$, $n = 0, \pm 1, \pm 2, \ldots$, such that $\inf_n y_n \leqq x \leqq \sup_n y_n$.

PROOF. The set of all x with the above property constitutes a σ-lattice containing L, and hence also $C(L)$.

If L is a lattice, $M(L)$ shall denote the set of all $x \in H$ with the property

$$(5.1) \qquad z \leqq y,\ z \in L,\ y \in L \Rightarrow (x \wedge y) \vee z = (x \vee z) \wedge y \in L\,.$$

The sign of equality follows from the distributivity (cf. [1, p. 133]). The element $(x \wedge y) \vee z = (x \vee z) \wedge y$ may be considered as a "middle point" between x, y, z. With the functional interpretation it becomes the "truncation" of x by y (above) and z (below). The requirement (5.1) is a possible definition of "measurable element" relatively to L (cf. [8]).

LEMMA 2. If L is a lattice over which a full integral is defined, then $C(L) \subset M(L)$.

PROOF. Evidently $L \subset M(L)$, and application of (5.1) and (I_4) proves $M(L)$ to be a σ-complete lattice.

THEOREM 6 (Uniqueness theorem). *Let I be an integral over L, and let $\tilde{I}$ be some full extension of I to $\tilde{L}$. Let $\tilde{L}_0$ be the intersection of all those subsets of $\tilde{L}$ containing L, to which the restrictions of $\tilde{I}$ have the Beppo Levi property (I_4). Then $\tilde{L}_0$ is a lattice, $\tilde{L}_0 = \tilde{L} \cap C(L)$, and the restriction of $\tilde{I}$ to $\tilde{L}_0$ is a unique, minimal, full extension of I within H.*

PROOF. 1) (The method of proof is taken from [5, p. 27–28].) We first notice that the restriction of $\tilde{I}$ to $\tilde{L}_0$ itself must have the Beppo Levi property, and thus $\tilde{L}_0$ is the smallest subset of $\tilde{L}$ containing L where $\tilde{I}$ has this property.

For each $x \in \tilde{L}_0$ we define $N(x)$ to be the set of all those $y \in \tilde{L}_0$ for which

$x \vee y \in \tilde{L}_0$, $x \wedge y \in \tilde{L}_0$. Then by the symmetry of the definition we have the equivalence $y \in N(x) \Leftrightarrow x \in N(y)$.

We are next to show that $N(x)$ must have the Beppo Levi property. Let $y_n \uparrow y$, $\sup_n (y_n) = \alpha < \infty$, $y_n \in N(x)$. Then $y_n \wedge x \in \tilde{L}_0$, and by the distributivity $y_n \wedge x \uparrow y \wedge x$. Since

$$\tilde{I}(y_n \wedge x) \leqq \tilde{I}(y_n) \leqq \alpha$$

for all n, and $\tilde{L}_0$ has the Beppo Levi property, we can conclude $y \wedge x \in \tilde{L}_0$. Similarly $y_n \vee x \in \tilde{L}_0$ and $y_n \vee x \uparrow y \vee x$. Since

$$\tilde{I}(y_n \vee x) = \tilde{I}(y_n) + \tilde{I}(x) - \tilde{I}(y_n \wedge x) \leqq \alpha + \tilde{I}(x) - \tilde{I}(y_1 \wedge x)$$

for all n, and $\tilde{L}_0$ has the Beppo Levi property, we can conclude $y \vee x \in \tilde{L}_0$. Hence both $y \wedge x \in \tilde{L}_0$ and $y \vee x \in \tilde{L}_0$, proving $y \in N(x)$.

The verification that $\tilde{L}_0$ is a lattice, will be accomplished if we can prove $\tilde{L}_0 \subset N(x)$ for every $x \in \tilde{L}_0$.

For $x \in L$ we have $L \subset N(x)$. Since $N(x)$ has the Beppo Levi property, this implies $\tilde{L}_0 \subset N(x)$ for all $x \in L$, and equivalently $L \subset N(y)$ for all $y \in \tilde{L}_0$, which again implies $\tilde{L}_0 \subset N(y)$ for all $y \in L_0$, q.e.d.

2) Application of the fact that $C(L)$ is a σ-lattice proves $C(L) \cap \tilde{L}$ to have the Beppo Levi property and hence $\tilde{L}_0 \subset C(L) \cap \tilde{L}$.

To prove the reverse inclusion, we first remark that the restriction of $\tilde{I}$ to $\tilde{L}_0$ must be a full integral since $\tilde{L}_0$ is a lattice and (I_4) is satisfied. Thus for every $x \in C(L) \cap \tilde{L}$ we can apply lemma 2 to obtain

$$x \in C(L) \subset C(\tilde{L}_0) \subset M(\tilde{L}_0) .$$

By lemma 1 there exist elements y_n of L such that $y_n \leqq y_{n+1}$, $n = 0, \pm 1, \pm 2, \ldots$ and $\inf_n y_n \leqq x \leqq \sup_n y_n$. The relation $x \in M(\tilde{L}_0)$ implies $(x \wedge y_n) \vee y_{-m} \in \tilde{L}_0$ for $m, n = 0, \pm 1, \pm 2 \ldots$. Hence for any fixed n

$$(x \wedge y_n) \vee y_{-m} \downarrow x \wedge y_n \qquad \text{as} \qquad m \to \infty ,$$

while

$$\tilde{I}\big((x \wedge y_n) \vee y_{-m}\big) \geqq \tilde{I}(x \wedge y_n) > -\infty .$$

Since $\tilde{L}_0$ has the Beppo Levi property, this implies $x \wedge y_n \in \tilde{L}_0$.

Repetition of the same procedure as $n \to \infty$ gives $x \wedge y_n \uparrow x$, while $\tilde{I}(x \wedge y_n) \leqq \tilde{I}(x) < \infty$ proving $x \in \tilde{L}_0$, q.e.d.

3) Let $\tilde{I}_1$ and $\tilde{I}_2$ be full extensions of I to $\tilde{L}_1$ and $\tilde{L}_2$, respectively. Let K be the set of those $x \in \tilde{L}_1 \cap \tilde{L}_2$ for which $\tilde{I}_1(x) = \tilde{I}_2(x)$. It is easily seen that K has the Beppo Levi property, and since $L \subset K$, we obtain by the first half of the theorem

$$\tilde{L}_1 \cap C(L) \subset K, \qquad \tilde{L}_2 \cap C(L) \subset K .$$

This shows that, for $x \in \tilde{L}_1 \cap C(L)$ and $x \in \tilde{L}_2 \cap C(L)$, we shall have $\tilde{I}_1(x) = \tilde{I}_2(x)$.

Moreover since $K \subset L_1$ and $K \subset L_2$, we obtain the following identities accomplishing the proof of the uniqueness theorem:

$$\tilde{L}_1 \cap C(L) = K \cap C(L) = \tilde{L}_2 \cap C(L) .$$

Remark. From th. 6 we obtain an interesting result of negative character, namely that it is impossible to extend I to a full integral defined for any other elements of $C(L)$ than those of $\tilde{L}_0$. In virtue of the extension theory of § 3 this means that it is impossible to extend I to *any* integral defined for elements of $C(L)$ not belonging to $\tilde{L}_0$. This is the general version of the well-known fact that it is impossible to extend the Riemann integral to any proper (i.e. finite-valued) integral defined for functions like $x(t) = t^{-1}$, etc.

Theorem 7. *Let I be an integral over L, and let $\tilde{I}_0$ defined over $\tilde{L}_0$ be the minimal, full extension of $\tilde{I}$. Then if I is some other full extension of I to a lattice $\tilde{L}$, such that L is dense in $\tilde{L}$ with respect to the pseudo-metric $\tilde{d}(x, y) = \tilde{I}(x \vee y) - \tilde{I}(x \wedge y)$, then to every $x \in \tilde{L}$ there may be assigned a $y \in \tilde{L}_0$ such that $\tilde{d}(x, y) = 0$.*

Proof. Let $x \in L$. Then we may find a sequence $\{x_n\}$, $x_n \in L$, for $n = 1, 2, \ldots$, such that $\tilde{d}(x_n, x_{n+1}) \leqq 2^{-n}$ and $\tilde{d}(x_n, x) \to 0$. By the lemma of § 1 we know that $y = \liminf_n x_n$ must exist in $\tilde{L}$ and that $\tilde{d}(x_n, y) \to 0$. Now

$$y \in C(L) \cap \tilde{L} = \tilde{L}_0$$

and by the triangel inequality $\tilde{d}(x, y) = 0$, q.e.d.

The result of theorem 7 applies in particular to the extensions obtained in § 3 and § 4. If I is an integral (or more generally, a valuation) over L, then we may assign to L a metric space obtained by identifying elements with zero pseudodistance. Theorem 7 states that the extensions of § 3 and § 4 give the same metric space as the minimal full extension. The passage from $\tilde{L}_0$ to the set $\tilde{L}$ of § 3 consists in an enlargement of the individual equivalence-classes; and so does the passage from the set $\tilde{L}$ to the set $\bar{L}$ of § 4, whenever the extension theory of § 4 is applicable (that is, when (I_3') is satisfied).

For certain applications it is useful to work with full integrals for which the equivalence classes are order-convex subsets of H. (Integrals of this kind correspond to the integrals defined by "complete measures" (cf. [5, p. 31].) Every full integral can be extended to a full integral with the above property simply by replacing the equivalence classes by their convex hulls. Clearly this is a unique minimal extension of the original full integral to a full integral with order-convex equivalence classes.

THEOREM 8. *The extended integral $\tilde{I}$ of § 3 can be obtained from the minimal full extension $\tilde{I}_0$ by passing from the equivalence classes to their convex hulls.*

PROOF. 1) From the definition of $\tilde{I}$ and $\tilde{L}$ it follows that the equivalence classes of this integral are order-convex.

2) Let $x \in \tilde{L}$. We are to prove that x belongs to the convex hull of some equivalence class of $\tilde{I}_0$. With use of the symbols from the proof of th. 4, we can find elements $y \in U^\circ$ and $z \in U_0$, such that $y \leqq x \leqq z$ and

$$I^\circ(y) - I_0(z) = \tilde{I}(y) - \tilde{I}(z) < \varepsilon .$$

Moreover we can find a descending sequence $\{y_n\}$ of such elements y and an ascending sequence $\{z_n\}$ of such elements z for which $\tilde{I}(y_n) - \tilde{I}(z_n) < 1/n$. Then, since

$$y_n \in C(L) \cap \tilde{L} = \tilde{L}_0, \quad z_n \in C(L) \cap \tilde{L} = \tilde{L}_0 ,$$

and $z_n \leqq x \leqq y_n$ for $n = 1, 2, \ldots$, we can conclude (by I_4) that

$$z_n \uparrow z \in \tilde{L}_0, \quad y_n \downarrow y \in \tilde{L}_0, \quad z \leqq x \leqq y, \quad \tilde{I}_0(z) = \tilde{I}_0(y) .$$

Hence x belongs to the order-convex hull of the common equivalence class of y and z of $\tilde{L}_0$, q.e.d.

Theorem 8 shows in particular that the values of the integral $\tilde{I}$ of § 3 for $x \in \tilde{L}$ are still uniquely determined. The values of the integral $\bar{I}$ of § 4 for $x \in \bar{L}$, however, are no longer unique. From the theory of integration in locally compact spaces we know that $\bar{I}$ becomes unique when an additional requirement, such as "regularity", is imposed (cf. [5, p. 239]).

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