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ON A THEOREM OF E. FØLNER

RALPH A. RAIMI

1. The purpose of this note is to prove anew a theorem of E. Følner [2, Theorem 2] concerning approximation properties of functions in a linear space without Banach mean value. The essential idea of the proof, even, is the same, but the argument is simplified by the observation that, in the case under consideration, the standard decomposition of a linear functional into the difference of positive parts preserves translation-invariance. Thus a more transparent form of the Hahn–Banach theorem can be used, and Følner's subadditive functional $\overline{M}$ need not be introduced. In the process, the non-essential role of the constant functions is put in evidence, and the hypothesis of their presence dropped from the theorem. The definition of *right Banach mean* is adjusted to make this possible; in the presence of the constant functions, the new definition is shown to coincide with that of Følner [3].

2. Let G be a group, L a linear space of bounded real-valued functions on G, with the norm

$$\|f\| = \sup\{|f(x)| \mid x \in G\} .$$

For each $a \in G$,

$$T_a\colon\ f(x) \to f(xa)$$

is called a right translation operator, and $\mathscr{T}$ will denote the collection $\{T_a \mid a \in G\}$. L is called *translation-invariant* if

$$f \in L,\ T \in \mathscr{T} \ \Rightarrow\ Tf \in L .$$

L is called *lattice-closed* if

$$f \in L,\ g \in L \ \Rightarrow\ \sup\{f, g\} \in L ,$$

where

$$\big(\sup\{f, g\}\big)(x) = \sup\{f(x), g(x)\} .$$

L^+ will denote the positive cone in L:

$$L^+ = \{f \in L \mid f \geqq 0\} ,$$

where $f \geqq 0$ means $f(x) \geqq 0$ for all $x \in G$. L^* will denote the conjugate space of L; if $\varphi \in L^*$ and $f \in L$, (φ, f) will denote the value of φ at f.

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We shall assume from now on that L is lattice-closed and translation-invariant. Let N be the closed linear extension of $\{Tf-f \mid T\in\mathscr{T}, f\in L\}$. A functional $\varphi\in L^*$ will be called a *right Banach mean* if it is non-zero and

(a) $(\varphi, f)\geqq 0$ for all $f\in L^+$,

(b) $(\varphi, f)=0$ for all $f\in N$.

We may say, then, that a right Banach mean is (a) positive, and (b) translation-invariant. If it happens that L contains the constant functions, in particular the 'unit-function' $u(x)=1$ for all $x\in G$, and if φ is a right Banach mean normalized so that $\|\varphi\|=1$, it follows from (a) that $(\varphi, u)=1$, by the following argument:

We know $(\varphi, u)\leqq 1$. If $|f(x)|\leqq 1$ for all $x\in G$, then $-u\leqq f\leqq u$, and $|(\varphi, f)|\leqq(\varphi, u)$ by (a). Since

$$\|\varphi\| = \sup\{|(\varphi, f)| \mid |f| \leqq u\},$$

we obtain $\|\varphi\|\leqq(\varphi, u)$, whence $(\varphi, u)=1$.

From this it follows that φ is a right Banach mean in the sense of Følner: Since

$$(\inf_x f(x))u \leqq f \leqq (\sup_x f(x))u,$$

therefore by (a) again,

$$\inf_x f(x) \leqq (\varphi, f) \leqq \sup_x f(x).$$

We will not, however, have to suppose in the sequel that $u\in L$.

3. Theorem (Følner). *If L has no right Banach mean, then $L=N$.*

Proof. If $L\neq N$, we shall construct a right Banach mean. Since N is a closed subspace of the normed space L, there exists an element $\varphi\in L^*$ which is non-zero, but vanishes on N. Since L is lattice-closed, φ may be decomposed into

$$\varphi = \varphi^+ - \varphi^-$$

by defining, for each $f\in L^+$,

$$(\varphi^+, f) = \sup\{(\varphi, g) \mid 0 \leqq g \leqq f\},$$

and then extending φ^+ to all of L by linearity (this procedure is described in [1, p. 35] and [2, p. 9]). Then φ^+ and φ^- are both positive functionals [1, p. 35]. Now φ^+ is continuous; indeed

$$\|\varphi^+\| \leqq \|\varphi\|,$$

by the following argument: If $f\in L^+$, it follows from the definition of φ^+ that

$$(\varphi^+, f) \leqq \|\varphi\|\cdot\|f\|.$$

For any $f \in L$, we have a decomposition $f = f^+ - f^-$, where $f^+ \in L^+$ and $f^- \in L^+$, by defining $f^+ = \sup\{f, 0\}$. Then

$$\|f^+\| \leqq \|f\| \quad \text{and} \quad \|f^-\| \leqq \|f\| .$$

Further

$$(\varphi^+, f) = (\varphi^+, f^+) - (\varphi^+, f^-) .$$

Hence

$$(\varphi^+, f) \leqq (\varphi^+, f^+) \leqq \|\varphi\| \cdot \|f^+\| \leqq \|\varphi\| \cdot \|f\|$$

and

$$(\varphi^+, f) \geqq -(\varphi^+, f^+) \geqq -\|\varphi\| \cdot \|f^-\| \geqq -\|\varphi\| \cdot \|f\| .$$

If we can show $(\varphi^+, T_a f - f) = 0$ for each $f \in L$ and each $a \in G$, then by the continuity of φ^+ it will follow that $(\varphi^+, f) = 0$ for all $f \in N$. Thus φ^+ and φ^- will both be positive and translation-invariant, and one of them at least will be the non-zero functional we seek. But

$$\begin{aligned}(\varphi^+, T_a f) &= \sup\{(\varphi, h) \mid 0 \leqq h \leqq T_a f\} \\ &= \sup\{(\varphi, T_{a^{-1}} h) \mid 0 \leqq T_a T_{a^{-1}} h \leqq T_a f\} \\ &= \sup\{(\varphi, T_{a^{-1}} h) \mid 0 \leqq T_{a^{-1}} h \leqq f\} \\ &= \sup\{(\varphi, h) \mid 0 \leqq h \leqq f\} \\ &= (\varphi^+, f) ,\end{aligned}$$

because of the translation-invariance of φ, and because (since G is a group) every $f \in L$ can be written in the form $T_{a^{-1}} h$, for a suitable $h \in L$.

Added in proof. It has come to the author's attention that Følner's theorem has already been stated and proved for the special case where L is the space of *all* bounded functions, in Corollaries $2'(d')$ and $5(d'')$ of: M. M. Day, *Means for the bounded functions and ergodicity of the bounded representations of semi-groups*, Trans. Am. Math. Soc. 69 (1950), 276–291.

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UNIVERSITY OF ROCHESTER, ROCHESTER, N. Y., U.S.A.