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ON THE EXPONENTIAL FORMULAS OF SEMI-GROUP THEORY

KAI LAI CHUNG[1]

1.

The purpose of this paper is to present a simple unified approach to a group of theorems in semi-group theory called the "exponential formulas", due to Hille, Phillips, Widder and D. G. Kendall [1, p. 354]. A more general and apparently new formula is arrived at, which includes some known cases. It turns out that these formulas are in essense summability methods which are best comprehended from the point of view of elementary probability theory. They are all in the spirit of S. Bernstein's proof of Weierstrass's approximation theorem, the same idea being present in M. Riesz's proof of Hille's first exponential formula (see [1, p. 314]). Whereas the details here are just a little simpler than in [1], it seems of some interest to exhibit the general pattern and to reduce the proofs to routine verifications. The reader who is not acquainted with the language of probability should have no difficulty in translating everything into the language of classical analysis. But the probability way of thinking is really germane to the subject.

2.

We employ the standard terminology of probability theory to be found in any introductory text, e.g. [2]. For any random variable (r.v.) β, $E(\beta)$ is its mean (or mathematical expectation), $V(\beta)$ is its variance:

$$V(\beta) = E\{[\beta - E(\beta)]^2\} = E(\beta^2) - [E(\beta)]^2 .$$

Integrals with respect to the basic probability measure will be denoted by $\int \ldots dP$. Thus $E(\beta) = \int_\Omega \beta \, dP$ where Ω is the whole probability space.

Let $\{\beta_\lambda\}$ be a family of (real-valued) r.v.'s indexed by λ, which may depend on a real parameter ξ. Finally, $g(t)$, $-\infty < t < \infty$, is a (strongly) measurable function of t with range in a Banach space, and $\|g(t)\|$ is its norm.

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LEMMA 1. *Suppose that*

(i) *as* $\lambda \to \infty$, β_λ *converges in probability to* ξ;
(ii) *for some* $r > 1$, $E\{\|g(\beta_\lambda)\|^r\}$ *is bounded in* λ;
(iii) $g(t)$ *is (strongly) continuous at* $t = \xi$.

Then we have

$$\lim_{\lambda \to \infty} E\{\|g(\beta_\lambda) - g(\xi)\|\} = 0 . \tag{1}$$

Furthermore, if $\{\beta_\lambda\}$ *depends on the parameter* ξ *and if the assumptions* (i), (ii) *and* (iii) *hold uniformly for* ξ *in a set, then the conclusion* (1) *also holds uniformly in this set.*

REMARK. (1) implies

$$g(\xi) = \lim_{\lambda \to \infty} E\{g(\beta_\lambda)\} .$$

In this form we must consider r.v.'s with range in a Banach space.

PROOF OF LEMMA 1. Although the lemma is a simple exercise in probability theory we give its proof here. For any $\delta > 0$,

$$E\{\|g(\beta_\lambda) - g(\xi)\|\} \leqq \left[\int_{|\beta_\lambda - \xi| \leqq \delta} + \int_{|\beta_\lambda - \xi| > \delta} \right] \|g(\beta_\lambda) - g(\xi)\| \, dP .$$

Given $\varepsilon > 0$, there exists by (iii) a $\delta > 0$ such that

$$\sup_{|t - \xi| \leqq \delta} \|g(t) - g(\xi)\| \leqq \varepsilon .$$

Having fixed δ, we have by Hölder's inequality:

$$\int_{|\beta_\lambda - \xi| > \delta} \|g(\beta_\lambda) - g(\xi)\| \, dP \leqq P\{|\beta_\lambda - \xi| > \delta\}^{(r-1)/r} E\{\|g(\beta_\lambda) - g(\xi)\|^r\}^{1/r}, \tag{2}$$

which tends to zero as $\lambda \to \infty$, by (i) and (ii). This proves (1). The uniformity supplement is obvious from the proof.

We can now recast M. Riesz's proof of Hille's first exponential formula as follows, treating here as in the sequel only the "strong case", since the "uniform case" is exactly similar.

THEOREM 1. *Let* $\{T(t), t \geqq 0\}$ *be a semi-group on* X *which is strongly continuous in* $t \in [\alpha, \infty)$, *and let* $\xi > \alpha \geqq 0$. *Then we have for every* $x \in X$:

$$T(\xi)x = \lim_{\eta \to 0} \exp\{(\xi - \alpha)(T(\eta) - I)/\eta\} \, T(\alpha)x , \tag{3}$$

uniformly for ξ *in any finite interval.*

PROOF. Let $\eta > 0$, and

$$\lambda = (\xi - \alpha)/\eta .$$

Let β_λ have a Poisson distribution $\pi(\alpha,\eta;\lambda)$, namely:

$$P\{\beta_\lambda = \alpha + n\eta\} = e^{-\lambda}\lambda^n/n!, \qquad n = 0,1,2,\ldots .$$

It is known from elementary probability theory that

$$E(\beta_\lambda) = \alpha + \lambda\eta = \xi ,$$
$$V(\beta_\lambda) = \lambda\eta^2 = (\xi-\alpha)\eta .$$

Hence we have by Chebyshev's inequality:

$$P\{|\beta_\lambda - \xi| > \delta\} \leqq V(\beta_\lambda)/\delta^2 = (\xi-\alpha)\eta/\delta^2$$

and consequently β_λ converges in probability to ξ as $\eta \to 0$ or $\lambda \to \infty$. For a fixed x in X, let

$$f(t) = T(t)x - T(\xi)x , \qquad t \geqq \alpha .$$

It is known from semi-group theory (see [1, p. 314]) that there exists an $M > 0$ such that $\|T(t)\| \leqq M^{1+t}$ for $t \geqq \alpha$. Hence

$$\|f(t)\|^2 \leqq 4M^2\|x\|^2 M^{2t} ,$$
$$E\{\|f(\beta_\lambda)\|^2\} \leqq 4M^2\|x\|^2 e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}M^{2(\alpha+n\eta)}$$
$$= 4M^{2+2\alpha}\|x\|^2\exp\{\lambda(M^{2\eta}-1)\} .$$

Since

$$\lim_{\eta\to 0}\lambda(M^{2\eta}-1) = (\xi-\alpha)\lim_{\eta\to 0}(M^{2\eta}-1)/\eta = 2(\xi-\alpha)\log M ,$$

$E\{\|f(\beta_\lambda)\|^2\}$ is bounded in λ, and uniformly so for ξ in any finite interval. Hence Lemma 1 is applicable. Since

$$E\{T(\beta_\lambda)x\} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}T(\eta)^n T(\alpha)x$$
$$= \exp\{-\lambda I + \lambda T(\eta)\}\, T(\alpha)x ,$$

(3) follows.

Next, we give a proof of D. G. Kendall's formula (see [1, p. 316]).

THEOREM 2. *Let* $\{T(t), t \geqq 0\}$ *be a semi-group on* X *which is strongly continuous in* $[0,1]$, *and* $T(0) = I$ *(identity). Then we have for every* $x \in X$:

$$T(\xi)x = \lim_{n\to\infty}\{(1-\xi)I + \xi T(1/n)\}^n x , \tag{4}$$

uniformly for ξ *in* $[0,1]$.

PROOF. Let $\{\beta_n, n \geqq 1\}$ be r.v.'s with the binomial distribution $B(n, \xi)$, namely:

$$P\{\beta_n = k\} = \binom{n}{k} \xi^k (1-\xi)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

It is known that

$$E(\beta_n/n) = \xi, \qquad V(\beta_n/n) = \xi(1-\xi)/n.$$

Hence β_n/n converges in probability to ξ as $n \to \infty$, uniformly for ξ in $[0,1]$. Furthermore, since $\|T\| \leqq M$ for some constant M by the uniform boundedness theorem, we have

$$E\{\|T(\beta_n/n)\|^2\} \leqq M^2.$$

Thus Lemma 1 is applicable. We have

$$\begin{aligned} E\{T(\beta_n/n)\,x\} &= \sum_{k=0}^{n} \binom{n}{k} (1-\xi)^{n-k} \xi^k T(k/n)\,x \\ &= \sum_{k=0}^{n} \binom{n}{k} ((1-\xi)I)^{n-k} (\xi T(1/n))^k x \\ &= \{(1-\xi)I + \xi T(1/n)\}^n x. \end{aligned}$$

Hence (4) follows.

3.

From now on let $\{T(t), t \geqq 0\}$ be a strongly continuous semi-group on X such that $\|T(t)\|$ is bounded as $t \downarrow 0$. Then there exist positive constants M and w such that

$$\|T(t)\| \leqq Me^{wt}, \qquad 0 \leqq t < \infty; \tag{5}$$

and the resolvent operator has the representation

$$R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\, dt, \qquad \lambda > w. \tag{6}$$

See [1, p. 321]. In fact, for what follows it will suffice to assume instead of (5) the strong continuity of $T(t)$ in $(0, \infty)$ and that $\int_0^1 \|T(t)x\|^r dt < \infty$ for some $r > 1$.

Let $e_\lambda, \lambda > 0$, be the exponential distribution function with parameter λ:

$$e_\lambda(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geqq 0, \\ 0, & t < 0. \end{cases}$$

Let $\{\tau_k(\lambda);\ k = 1, 2, \ldots\}$ be independent r.v.'s all having the distribution function e_λ, and let

$$\sigma_0(\lambda) = 0\,, \qquad \sigma_n(\lambda) = \sum_{k=1}^{n} \tau_k(\lambda)\,, \qquad n = 1,2,\ldots\,. \tag{7}$$

It is known that

$$\begin{aligned} E(\tau_k(\lambda)) &= 1/\lambda\,, \qquad V(\tau_k(\lambda)) = 1/\lambda^2; \\ E(\sigma_n(\lambda)) &= n/\lambda\,, \qquad V(\sigma_n(\lambda)) = n/\lambda^2. \end{aligned} \tag{8}$$

LEMMA 2. *For $\lambda > w$, we have*

$$E\{T(\sigma_n(\lambda))\} = [\lambda R(\lambda)]^n\,.$$

PROOF. Clearly for each k,

$$E\{T(\tau_k(\lambda))\} = \int_0^\infty \lambda e^{-\lambda t} T(t)\,dt = \lambda R(\lambda)\,.$$

Since the $\tau_k(\lambda)$'s are independent, so are the $T(\tau_k(\lambda))$'s as operator-valued r.v.'s, and we have

$$E\{T(\sigma_n(\lambda))\} = E\left\{\prod_{k=1}^{n} T(\tau_k(\lambda))\right\} = \prod_{k=1}^{n} E\{T(\tau_k(\lambda))\} = [\lambda R(\lambda)]^n\,.$$

We give an alternative proof. It is known, and easily shown by induction on n that

$$P\{\sigma_n(\lambda) \leqq t\} = \int_0^t \frac{(\lambda s)^{n-1}}{(n-1)!}\,\lambda e^{-\lambda s}\,ds\,, \qquad t \geqq 0\,. \tag{9}$$

Hence we have

$$\begin{aligned} E\{T(\sigma_n(\lambda))\} &= \int_0^\infty \frac{(\lambda t)^{n-1}}{(n-1)!}\,\lambda e^{-\lambda t} T(t)\,dt \\ &= \frac{(-1)^{n-1}\lambda^n R^{(n-1)}(\lambda)}{(n-1)!} = [\lambda R(\lambda)]^n\,, \end{aligned}$$

the last equation being a property of the resolvent operator.

The next theorem is the semi-group version of the Post–Widder theorem [1, p. 224, p. 352].

THEOREM 3. *We have for every $\xi > 0$, and $x \in X$:*

$$T(\xi)x = \lim_{n\to\infty} [(n/\xi)R(n/\xi)]^n x\,,$$

uniformly for ξ in any finite interval.

PROOF. For each n, let τ_{nk}, $1 \leqq k \leqq n$, be independent r.v.'s all having the distribution function e_λ, with $\lambda = n/\xi$; and let $\sigma_n = \sum_{k=1}^{n} \tau_{nk}$. Then by (8)

$$E(\sigma_n) = \xi\,, \qquad V(\sigma_n) = \xi^2/n\,.$$

Hence σ_n converges to ξ probability as $n \to \infty$, uniformly for ξ in any finite interval. Next, by (5) and (9),

$$
\begin{aligned}
E\{\|T(\sigma_n)x\|^2\} &= \int_0^\infty \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} \|T(t)x\|^2 \, dt \\
&= \int_0^\infty \frac{s^{n-1}}{(n-1)!} e^{-s} \|T(s/\lambda)x\|^2 \, ds \\
&\leqq \frac{M^2\|x\|^2}{(n-1)!} \int_0^\infty s^{n-1} e^{-(1-2w/\lambda)s} \, ds \\
&= M^2 \|x\|^2 (1 - 2w\xi/n)^{-n} .
\end{aligned}
$$

This is bounded in n, uniformly for ξ in any finite interval. Hence Lemma 1 is applicable and we have by Lemma 2,

$$
T(\xi)x = \lim_{n\to\infty} E\{T(\sigma_n)x\} = \lim_{n\to\infty} [(n/\xi)R(n/\xi)]^n x .
$$

Remark. The original Widder theorem for a numerically valued function g is sharper than Theorem 3 in two respects: (1) g is assumed only to be (substantially) in L^1, (2) the conclusion holds for ξ in the Lebesgue set of g. Our method here applies when g is in L^r for some $r > 1$; cf. condition (ii) of Lemma 1. This is an essentially stronger uniform integrability condition and accounts largely for the simplification of proof. The conclusion for a Lebesgue set under this assumption, however, can be easily obtained by calculating the maximum probability density of σ_n. Similar remarks apply to Phillips's Theorem 4 below.

4.

The next probability concept needed is that of the sum of a random number of random variables, but in a rather trivial form.

Lemma 3. *Let $\{\tau_k, k \geqq 1\}$ be independent, identically distributed r.v.'s, $\sigma_n = \sum_{k=1}^n \tau_k$; and let ν be a non-negative integer-valued r.v. independent of the τ_k's. If τ_1 and ν have finite variances, then*

$$
\begin{aligned}
E(\sigma_\nu) &= E(\nu)E(\tau_1) , \\
V(\sigma_\nu) &= E(\nu)V(\tau_1) + V(\nu)E(\tau_1)^2 .
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
E(\sigma_\nu) &= \sum_{n=0}^{\infty} P\{\nu=n\}\, E(\sigma_n) = \sum_{n=0}^{\infty} P\{\nu=n\}\, n E(\tau_1) = E(\nu) E(\tau_1); \\
V(\sigma_\nu) &= E(\sigma_\nu^2) - E(\sigma_\nu)^2 = \sum_{n=0}^{\infty} P\{\nu=n\}\, E(\sigma_n^2) - E(\sigma_\nu)^2 \\
&= \sum_{n=0}^{\infty} P\{\nu=n\}\, \{V(\sigma_n) + E(\sigma_n)^2\} - E(\sigma_\nu)^2 \\
&= \sum_{n=0}^{\infty} P\{\nu=n\} \{n V(\tau_1) + n^2 E(\tau_1)^2\} - E(\nu)^2 E(\tau_1)^2 \\
&= E(\nu) V(\tau_1) + \{E(\nu^2) - E(\nu)^2\} E(\tau_1)^2 \\
&= E(\nu) V(\tau_1) + V(\nu) E(\tau_1)^2 .
\end{aligned}
$$

The following theorem is due to Phillips [1, p. 221, p. 351].

THEOREM 4. *We have for every* $\xi > 0$, *and* $x \in X$:

$$
T(\xi) x = \lim_{\lambda \to \infty} \exp\{\xi[\lambda^2 R(\lambda) - \lambda I]\} x , \tag{10}
$$

uniformly for ξ *in any finite interval.*

PROOF. Fix $\xi > 0$; for each $\lambda > 0$, let $\nu = \nu(\lambda)$ have the Poisson distribution $\pi(0, 1; \lambda\xi)$:

$$
P\{\nu = n\} = e^{-\lambda\xi} (\lambda\xi)^n / n! , \qquad n = 0, 1, 2, \dots .
$$

We have $E(\nu) = \lambda\xi$, $V(\nu) = \lambda\xi$. Let the τ_k's and σ_n's be as in (7) and let ν be independent of them. We write σ_ν simply for $\sigma_{\nu(\lambda)}(\lambda)$. It follows from (8) and Lemma 3 that

$$
\begin{aligned}
E(\sigma_\nu) &= \lambda\xi \cdot (1/\lambda) = \xi , \\
V(\sigma_\nu) &= \lambda\xi \cdot (1/\lambda^2) + \lambda\xi \cdot (1/\lambda^2) = 2\xi/\lambda .
\end{aligned}
$$

Hence again by Chebyshev's inequality, σ_ν converges to ξ in probability as $\lambda \to \infty$, uniformly for ξ in any finite interval. Furthermore,

$$
\begin{aligned}
E\{\|T(\sigma_\nu) x\|^2\} &= e^{-\lambda\xi} \sum_{n=0}^{\infty} \frac{(\lambda\xi)^n}{n!} E\{\|T(\sigma_n) x\|^2\} \\
&= e^{-\lambda\xi} \sum_{n=0}^{\infty} \frac{(\lambda\xi)^n}{n!} \int_0^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} \|T(t) x\|^2 \, dt .
\end{aligned}
$$

By (5), the last integral is bounded by

$$
\int_0^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{(-\lambda + 2w)t} \lambda M^2 \|x\|^2 \, dt = M^2 \|x\|^2 \left(\frac{\lambda}{\lambda - 2w} \right)^n \tag{11}
$$

provided $\lambda > 2w$. Hence

$$E\{\|T(\sigma_\nu)x\|^2\} \leqq M^2\|x\|^2 \exp\left\{-\lambda\xi + \frac{\lambda^2\xi}{\lambda-2w}\right\} = M^2\|x\|^2 \exp\left\{\frac{2\lambda w\xi}{\lambda-2w}\right\}.$$

This is bounded in λ, uniformly for ξ in any finite interval. Hence Lemma 1 is applicable. We have by Lemma 2,

$$\begin{aligned} E\{T(\sigma_\nu)x\} &= e^{-\lambda\xi}\sum_{n=0}^{\infty}\frac{(\lambda\xi)^n}{n!}E\{T(\sigma_n)x\} \\ &= e^{-\lambda\xi}T(0)x + e^{-\lambda\xi}\sum_{n=1}^{\infty}\frac{(\lambda\xi)^n}{n!}[\lambda R(\lambda)]^n x \\ &= e^{-\lambda\xi}T(0)x + e^{-\lambda\xi}\{e^{\lambda^2 R(\lambda)\xi} - I\}x\,. \end{aligned}$$

Thus (10) follows.

Remark. We note that the Laplace transform of σ_ν is

$$E\{e^{-u\sigma_\nu}\} = \exp\left\{\frac{-\lambda\xi u}{\lambda+u}\right\}.$$

As $\lambda \to \infty$ this tends to $e^{-\xi u}$, showing again that σ_ν converges to ξ in probability.

5.

It is natural to replace the Poisson r.v. ν in the proof of Theorem 4 by a more general one. We content ourselves with the following theorem, though more sophisticated results can be obtained by using a continuous r.v. or relaxing the condition (12) below. We can even use other than the exponential distribution for the τ's in (7), which is dictated only by the pre-eminence of the resolvent.

Theorem 5. *Let $\xi \geqq 0$, and*

$$\varphi_\xi(u) = \sum_{k=0}^{\infty} a_k(\xi)u^k\,,$$

where

$$a_k(\xi) \geqq 0\,, \qquad \sum_{k=0}^{\infty} a_k(\xi) = 1, \qquad \sum_{k=0}^{\infty} k\,a_k(\xi) = \xi\,.$$

Suppose that we have for some $\varrho_\xi > e^{2w}$, where w is as in (5),

$$\varphi_\xi(\varrho_\xi) < \infty\,. \tag{12}$$

Then if the semi-group T satisfies the condition $T(0)=I$ in addition to the assumptions made at the beginning of § 3, *we have for every $x \in X$:*

$$T(\xi)x = \lim_{n\to\infty}\left[\varphi_\xi(nR(n))\right]^n x\,. \tag{13}$$

If the ϱ_ξ *in* (12) *can be chosen independently of* ξ *in every finite interval, then the convergence in* (13) *is uniform for* ξ *in every finite interval.*

PROOF. Let $\{\eta_k, k \geqq 1\}$ be independent, identically distributed, non-negative integer-valued r.v.'s all having φ_ξ as generating function, namely:

$$E\{u^{\eta_k}\} = \varphi_\xi(u) .$$

Let $\zeta_n = \sum_{k=1}^n \eta_k$; then the generating function of ζ_n is

$$E\{u^{\zeta_n}\} = \sum_{k=0}^{\infty} a_k^{(n)} u^k = [\varphi_\xi(u)]^n ,$$

where

$$a_k^{(n)} = P\{\zeta_n = k\} .$$

Let the τ_k's all have the distribution e_n and let the τ's and η's be altogether independent. It follows from (12) that η_k has a finite variance v_ξ^2 and

$$E(\zeta_n) = n\xi , \qquad V(\zeta_n) = n v_\xi^2 .$$

Hence by Lemma 3, and (8):

$$E(\sigma_{\zeta_n}) = n\xi\cdot(1/n) = \xi ,$$
$$V(\sigma_{\zeta_n}) = n\xi\cdot(1/n^2) + n v_\xi^2\cdot(1/n^2) = (\xi + v_\xi^2)/n .$$

Consequently σ_{ζ_n} converges in probability to ξ as $n \to \infty$. Using (11), we have

$$\begin{aligned} E\{\|T(\sigma_{\zeta_n})x\|^2\} &= \sum_{k=0}^{\infty} a_k^{(n)} E\{\|T(\sigma_k)x\|^2\} \\ &\leqq M^2\|x\|^2 \sum_{k=0}^{\infty} a_k^{(n)} \left(\frac{n}{n-2w}\right)^k = M^2\|x\|^2 \left[\varphi_\xi\left(\frac{n}{n-2w}\right)\right]^n . \end{aligned}$$

By Hölder's inequality, we have

$$[\varphi_\xi(u)]^n \leqq \sum_{k=0}^{\infty} a_k(\xi) u^{nk} .$$

(This is also known as Liapounov's inequality.) Hence we have, for each $\varrho > e^{2w}$, as soon as n is so large that $|n/(n-2w)|^n < \varrho$,

$$\left[\varphi_\xi\left(\frac{n}{n-2w}\right)\right]^n \leqq \sum_{k=0}^{\infty} a_k(\xi)\varrho^k .$$

Consequently this quantity is bounded in n by (12), and so is $E\{\|T(\sigma_{\zeta_n})x\|^2\}$, uniformly for ξ in any finite interval if the ϱ_ξ in (12) can be chosen independently of ξ in the same interval. In this case v_ξ^2 will be uniformly bounded in ξ and the convergence of σ_{ζ_n} to ξ is likewise uniform. Therefore Lemma 1 is applicable. Furthermore, since

$$\|[nR(n)]^k x\| \leqq M\|x\|\left(\frac{n}{n-w}\right)^k,$$

$\varphi_\xi(nR(n))$ is a bounded operator for all sufficiently large n. We have by Lemma 2,

$$E\{T(\sigma_{\zeta_n})x\} = \sum_{k=0}^{\infty} a_k^{(n)} E\{T(\sigma_k)x\}$$

$$= \sum_{k=0}^{\infty} a_k^{(n)} [nR(n)]^k x = [\varphi_\xi(nR(n))]^n x .$$

Thus (13) follows.

PARTICULAR CASES.

(i) $\varphi_\xi(u) = 1-\xi+\xi u, \quad 0 \leqq \xi \leqq 1;$

$$T(\xi)x = \lim_{n\to\infty} [(1-\xi)I+\xi nR(n)]^n x .$$

This is to be compared with Theorem 2 above. The semi-group property of T can be used to extrapolate this formula to all $\xi \geqq 0$, in various obvious ways.

(ii) $\varphi_\xi(u) = \dfrac{1}{1+\xi}\displaystyle\sum_{n=0}^{\infty}\left(\frac{\xi}{1+\xi}\right)^n u^n = \frac{1}{1+\xi-\xi u};$

$$T(\xi)x = \lim_{n\to\infty} [(1+\xi)I-\xi nR(n)]^{-n} x = \lim_{n\to\infty} \left[I-\xi(nR(n)-I)\right]^{-n} x ,$$

provided $w=0$.

(iii) $\varphi_\xi(u) = e^{-\xi} e^{\xi u} = \displaystyle\sum_{n=0}^{\infty} e^{-\xi}\frac{\xi^n}{n!}u^n;$

$$\mathrm{T}(\xi)x = \lim_{n\to\infty} \exp\{-\xi nI+\xi n^2 R(n)\} x .$$

This is Theorem 4 for integer values of λ.

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STANFORD UNIVERSITY, STANFORD, CALIF., U.S.A.