THE BRAID GROUPS

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1. Introduction.

The braid groups $B_n$, $n = 1, 2, 3, \ldots$, were introduced in 1926 by E. Artin [1] and have been the subject of numerous investigations. Although there is a well-known presentation of $B_n$ that has been derived several times the derivations that appear in the literature e.g. [1], [2] are all, in one way or another, somewhat devious. Our principal object is to give a straightforward derivation of this presentation, based on the previously unnoted remark that $B_n$ may be considered as the fundamental group of the space $E^{2n}$ of configurations of $n$ undifferentiated points in the plane.

Our derivation uses a method of computation that has never been published, although knowledge of it is probably widely distributed. It is proposed to publish the details of this method in a later paper; however the ideas involved are transparent enough to be believably communicated very briefly, and this we do in § 2 of the present paper.

By examining a certain covering of $E^{2n}$ and using the results of [3] it is shown that $E^{2n}$ is aspherical, and certain consequences of this fact are noted. In particular it follows immediately that $B_n$ has no elements of finite order; we believe that this was not previously known.

2. Computation of $\pi_1$.

If $X$ is a regular cell-complex, then we consider mappings of $X$ onto $X/R$ where $R$ is a relation obtained from a family $\Phi$ of identifications of the cells of $X$. $\Phi$ is required to satisfy the following conditions:

0) Each $\varphi$ in $\Phi$ is a homeomorphism with domain a closed cell of $X$.
   i) If $U$ is a cell of $X$, $\varphi: \overline{U} \to \overline{U}$ is in $\Phi$ if and only if $\varphi$ is the identity.
   ii) If $\varphi \in \Phi$, $\varphi: \overline{U}_1 \to \overline{U}_2$ then $\varphi^{-1}: \overline{U}_2 \to \overline{U}_1$ is in $\Phi$.
   iii) If $\varphi: \overline{U}_1 \to \overline{U}_2$ and $\varphi^1: \overline{U}_2 \to \overline{U}_3$ are in $\Phi$, so is $\varphi^1 \varphi: \overline{U}_1 \to \overline{U}_3$.
   iv) If $\varphi: \overline{U}_1 \to \overline{U}_2$ is in $\Phi$ and $V_1$ is a cell contained in $\overline{U}_1$ then $V_2 = \varphi(V_1)$ is also a cell, and $\overline{V}_1: \overline{V}_1 \to \overline{V}_2$ is in $\Phi$.

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In what follows $X$ and $X/R$ will be manifolds of dimension $n$, and we shall compute $\pi_1$ of the complement of an $(n - 2)$-dimensional subcomplex $K$ of $X/R$.

The algorithm for the computation is roughly as follows: Select in $X/R$ a maximal $n$-dimensional “cave” $C$ of $n$-dimensional, and oriented $(n - 1)$-dimensional cells (this will be dual to a maximal tree in a dual cell decomposition). To each oriented $(n - 1)$-cell not in the cave will correspond a generator of $\pi_1$. This generator is represented by a loop that penetrates the $(n - 1)$ cell once with intersection number 1 but otherwise lies entirely in $C$. To each $(n - 2)$-cell of $X/R$ that does not belong to $K$ will correspond a relation, obtained from the “non-abelian coboundary” of the $(n - 2)$-cell in question. More precisely, a small loop about an $(n - 2)$-cell $\sigma$ will intersect, in a certain order and sense, all the $(n - 1)$-cells having $\sigma$ on their boundary. Joining this loop to the base point will give a representative of an element of the fundamental group of the union of the $n$, and $(n - 1)$-cells of $X/R - K$. In this way a set of elements of the free group generated by the $(n - 1)$-cells not in $C$ is defined. This set of elements, one for each $(n - 2)$-cell not in $K$, will be a complete set of relations for $\pi_1(X/R - K)$.

3. A cellular decomposition of $S^{2n}$.

An ordered $n$-tuple $(p_1, \ldots, p_n)$ of points of the plane $E^2$ may be considered to be a point $p$ of $2n$-dimensional space $E^{2n}$. If the coordinates of $p_i$ are $x_i, y_i$, the coordinates of the corresponding point $p$ are

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n.$$

Let us write $i_1 < i_2$ whenever the abscissa of $p_{i_1}$ is smaller than the abscissa of $p_{i_2}$, $i_1 \nsucceq i_2$ whenever $p_{i_1}$ and $p_{i_2}$ have the same abscissa, and the ordinate of $p_{i_1}$ is smaller than the ordinate of $p_{i_2}$, and $i_1 = i_2$ whenever $p_{i_1}$ coincides with $p_{i_2}$. Information of this sort can be condensed into a single symbol, $\theta$, describing a point set in $E^{2n}$. Thus, for example, the symbol $(3 < 5 = 1 < 6 \nsucceq 4 \nsucceq 2 = 7)$ denotes the set of all points $(x_1, y_1, \ldots, x_7, y_7)$ in $E^{14}$ such that

$$x_3 < x_5 = x_1 < x_6 = x_4 = x_2 = x_7,$$

$$y_5 = y_1, \quad y_6 < y_4 < y_2 = y_7.$$

(Of course the same information is indicated by each of the symbols

$$(3 < 1 = 5 < 6 \nsucceq 4 \nsucceq 2 = 7),$$
$$(3 < 5 = 1 < 6 \nsucceq 4 \nsucceq 7 = 2),$$
$$(3 < 1 = 5 < 6 \nsucceq 4 \nsucceq 7 = 2);$$
we shall not distinguish between such equivalent symbols). The same symbol \( \theta \) will be used to denote the set of all those points \( p \) satisfying the indicated conditions.

It is easy to see that each \( \theta \) is a convex subset of \( E^{2n} \) and that, together with the point at infinity, they are the (open) cells of a regular cell-subdivision of the 2n-dimensional sphere \( S^{2n} = E^{2n} \cup \infty \). The dimension of the cell \( \theta \) is obviously equal to \( 2n \) minus the sum of the number of occurrences of \( \vee \) and twice the number of occurrences of \( = \). The lower dimensional cells that are on the boundary of \( \theta \) are obtained by replacing instances of \( i_1 < i_2 \) by \( i_1 \vee i_2 \) or \( i_2 \vee i_1 \) and/or replacing instances of \( j_1 \not\leq j_2 \) by \( j_1 = j_2 \) (or \( j_2 = j_1 \)). For example the boundary of the 5-dimensional cell \( (1 < 2 \vee 3) \) consists of the 4-dimensional cells \( (1 \vee 2 \not\leq 3), (2 \not\leq 1 \vee 3), (1 < 2 = 3) \), the 3-dimensional cells \( (1 = 2 \not\leq 3), (1 \vee 2 = 3), (2 \not\leq 1 = 3) \), the 2-dimensional cell \( (1 = 2 = 3) \), and the vertex \( \infty \).

In what follows we shall be concerned especially with the cells of dimension \( \geq 2n - 2 \). There are \( n! \) cells of dimension \( 2n \). One of them is \( (1 < 2 < \ldots < n) \), and the others may be obtained from this by permuting the indices \( 1, 2, \ldots, n \). The \( (2n - 1) \)-cells on the boundary of

\[
(1 < 2 < \ldots < n)
\]

are

\[
(1 \vee 2 < 3 < \ldots < n),
(2 \vee 1 < 3 < \ldots < n),
(1 < 2 \not\leq 3 < \ldots < n),
(1 < 3 \not\leq 2 < \ldots < 2n)
\text{ etc.,}
\]

and the \( (2n - 2) \)-cells on the boundary of, say, \( (1 \not\leq 2 < 3 < \ldots < n) \) are

\[
(1 = 2 < 3 < \ldots < n),
(1 \vee 2 \not\leq 3 < \ldots < n),
(1 \not\leq 3 \not\leq 2 < \ldots < n),
(3 \not\leq 1 \not\leq 2 < \ldots < n),
(1 \not\leq 2 < 3 \not\leq 4 < \ldots < n),
(1 \not\leq 2 < 4 \not\leq 3 < \ldots < n)
\text{ etc.}
\]

4. The action of \( \Sigma_n \) on \( S^{2n} \).

To the permutation

\[
\begin{pmatrix}
1 & 2 & \ldots & n \\
i_1 & i_2 & \ldots & i_n
\end{pmatrix}
\]

associate the autohomeomorphism of \( S^{2n} \) that maps an arbitrary point
(p_1, p_2, \ldots, p_n) of E^{2n} into the point (p_{i1}, p_{i2}, \ldots, p_{in}); thus an action on S^{2n} of the symmetric group \Sigma_n of permutations of n symbols 1, 2, \ldots, n is defined. Denote the collapsed space by \hat{S}^{2n}, and the image of E^{2n} under the collapsing \Lambda by \hat{E}^{2n}. Each of the autohomeomorphisms of S^{2n} considered maps \infty into \infty and permutes the cells \theta; the collapsing \Lambda maps one or more m-cells \sigma upon an m-cell \tau of \hat{E}^{2n}, (not necessarily homeomorphically). The cells \tau, together with the image of the point at \infty, constitute a regular cell-subdivision with identifications of \hat{S}^{2n}. A symbolic designation of the cells \tau is readily derived. For example the cells of \hat{S}^2 are \{1 < 2 < 3\}, \{1 < 2 \lor 3\}, \{1 \lor 2 < 3\}, \{1 \lor 2 \lor 3\}, \{1 < 2 = 3\}, \{1 = 2 < 3\}, \{1 \lor 2 = 3\}, \{1 = 2 \lor 3\}, \{1 = 2 = 3\}, and \infty.

5. The subcomplex \Lambda.

The points p_1, \ldots, p_n of E^2 are distinct if and only if, for each i < j, (x_i - x_j)^2 + (y_i - y_j)^2 > 0. Accordingly we consider the collection \Lambda of those cells \theta of our decomposition of E^{2n} in whose symbols the sign = occurs at least once. Since boundaries are obtained by changing < to \lor or \lor to =, it is clear that \Lambda and \infty together form a (2n - 2)-dimensional subcomplex of the cell complex S^{2n}. Furthermore the points p_1, \ldots, p_n of E^{2n} are distinct if and only if p lies in E^{2n} - \Lambda. Let \hat{\Lambda} denote the image of \Lambda under the collapsing \Lambda of S^{2n} to \hat{S}^{2n}. Then \hat{\Lambda} \cup \infty is a subcomplex of the cell complex \hat{S}^{2n}, and p_1, \ldots, p_n are distinct if and only if \hat{p} \in \hat{E}^{2n} - \hat{\Lambda}. Note that the point \hat{p} may be considered to be an unordered n-tuple of points p_1, \ldots, p_n of E^2. Let E^{2n} = E^{2n} - \hat{\Lambda}.

6. The Braid group.

Let B_n denote the braid group on n strings, \varphi the well-known homomorphism of B^n upon \Sigma^n, and \mathcal{I}^n the kernel of \varphi. If we look at the plane cross sections of a braid \mathcal{G}, we see that it may be described kinematically as a motion of n distinct points in the plane that ends with these points back in their original position but permuted as indicated by the permutation \varphi(\mathcal{G}). In particular \mathcal{G} belongs to \mathcal{I}_n if and only if the motion described returns each point to its original position. From these remarks it should be clear that the fundamental group of E^{2n} - \Lambda is \mathcal{I}_n, the fundamental group of \hat{E}^{2n} - \hat{\Lambda} is B_n, and that E^{2n} - \Lambda is the unbranched covering space of \hat{E}^{2n} - \hat{\Lambda} that belongs to the subgroup \mathcal{I}_n of B_n.

7. A presentation of B_n.

To calculate \pi_1(\hat{E}^{2n} - \hat{\Lambda}) choose the base point in the interior of the 2n-cell \lambda^{2n} = \{1 < 2 < \ldots < n\}. Since this is the only 2n-cell of \hat{S}^{2n}, there is a generator \sigma_j corresponding to each (2n - 1)-cell
\[ \lambda_j^{2n-1} = \{ \ldots < j \geq j+1 < \ldots \}; \]

it is represented by a loop in \( \lambda^{2n} \cup \lambda_j^{2n-1} \) that cuts \( \lambda_j^{2n-1} \) exactly once. Let us suppose that \( \lambda_j^{2n-1} \) is so oriented that the motion of \( p_1 \cup \ldots \cup p_n \) in \( E^2 \) described by a loop representative of \( \sigma_j \) causes the points \( p_j \) and \( p_{j+1} \) to interchange places (and names) by circling one another in a counterclockwise direction. The motion for \( \sigma_j^{-1} \) is shown in Figure 1.

![Fig. 1.](image)

The braid \( \sigma_j^{-1} \) is shown in Figure 2.

![Fig. 2.](image)

According to the general theory, a complete set of relations can be found in one to one correspondence with the cells of \( E^{2n} - \hat{A} \) of dimension \( 2n-2 \). These are of two sorts:

\[ \lambda_{i,k} = \{ \ldots < i \geq i+1 < \ldots < k \geq k+1 < \ldots \}, \quad i+1 < k, \]

\[ \lambda_{i, i+1} = \{ \ldots < i \geq i+1 \geq i+2 < \ldots \}. \]

Now \( \lambda_{i,k} \) is on the boundary of just the \((2n-1)\)-cells \( \lambda_i \) and \( \lambda_k \). Figure 3 shows a local cross section of \( E^{2n} \) by a plane perpendicular to the \((2n-2)\)-cell

\[ (1 < \ldots < i \geq i+1 < \ldots < k \geq k+1 < \ldots < n). \]

![Fig. 3.](image)
The relation \( r_{i,k} \) corresponding to the cell \( \lambda_{i,k} \) in \( \hat{E}^{2n} \) is read as a "non-abelian coboundary" of \( \lambda_{i,k} \). It is

\[
 r_{i,k} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}
\]

as may be seen by traversing the dotted loop in Figure 3. The motion of \((p_1, \ldots, p_n)\) in \( E^2 \) described by \( r_{i,k} \) is shown in Figure 4 and its interpretation as a braid in Figure 5.

![Figure 4](image)

![Figure 5](image)

As for \( \lambda_{i, i+1} \), it is on the boundary of \( \lambda_i \) and \( \lambda_{i+1} \). A local cross section of \( E^{2n} \) by a plane perpendicular to the \((2n-2)\)-cell

\[
(1 < \ldots < i \nleq i+1 \nleq i+2 < \ldots < n)
\]

is shown in Figure 6.
The corresponding relation \( r_{i, i+1} \) is
\[
r_{i, i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}
\]
as may be seen by traversing the dotted loop in Figure 6. The motion of \((p_1, \ldots, p_n)\) in \(E^2\) thereby described is shown in Figure 7, and its interpretation as a braid in Figure 8.

Thus we have derived anew the well-known presentation
\[
\mathcal{B}_n = (\sigma_1, \ldots, \sigma_{n-1} : r_{1,2}, r_{1,3}, \ldots, r_{n-2,n-1}).
\]

Remark. The same method could be used to find a presentation of \(\mathcal{F}_n\), but the result could just as well be obtained by applying the Reide-meister–Schreier theorem.

8. Corollaries.

The covering of \(\tilde{E}^{2n}\) corresponding to the representation of \(\mathcal{B}_n\) on \(\Sigma_n\) (symmetric group of degree \(n\)) is just the space \(F_{0,n}^2\) of [3], hence according to [3] has trivial homotopy groups above dimension 1. It follows then that \(\tilde{E}^{2n}\) is aspherical. As an immediate corollary we have:

Corollary 1. \(\mathcal{B}_n\) has no elements of finite order.

Proof. \(\tilde{E}^{2n}\) is a finite dimensional \(K(\mathcal{B}_n, 1)\) space, hence every subgroup of \(\mathcal{B}_n\) must be of finite geometric, hence finite cohomological dimension, but an element of finite order would generate a subgroup of infinite cohomological dimension.

Clearly \(\tilde{E}^{2n}\) is an open \(2n\)-dimensional manifold so we have:

Corollary 2. \(\mathcal{B}_n\) has the homology groups of an open \(2n\)-dimensional manifold.

Remark. It seems reasonable to expect that the homology groups of \(\tilde{E}^{2n}\), which by virtue of the asphericity of \(\tilde{E}^{2n}\) are those of \(\mathcal{B}_n\), may be calculated from the cellular decomposition of \(\tilde{E}^{2n}\) which we have given.

BIBLIOGRAPHY