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ON THE COMPARISON THEOREM OF KNESER–HILLE

AUREL WINTNER

1. This note deals with a criterion for the oscillatory or disconjugate character of a differential equation

$$x''+f(t)x = 0 \tag{1}$$

on a half-line const. $<t<\infty$, when (1) is compared with a differential equation of the same form, say with

$$y''+g(t)y = 0 \tag{2}$$

(concerning the nomenclature, and for a presentation of some of the known results, cf. [1, 227–234]).

For large positive t, let $f(t)$, $g(t)$ be a pair of real-valued, continuous functions for which the integrals

$$\int^{\infty} f(t)\,dt, \quad \int^{\infty} g(t)\,dt, \qquad \text{where} \qquad \int^{\infty} = \lim_{T\to\infty}\int^{T}, \tag{3}$$

are convergent (possibly just conditionally). Suppose further that the integrals (3), when considered as functions,

$$F(t) = \int_t^{\infty} f(s)\,ds, \qquad G(t) = \int_t^{\infty} g(s)\,ds\,, \tag{4}$$

of the lower limit of integration, satisfy the inequalities

$$0 \leqq G(t) \leqq F(t) \tag{5}$$

(for large t). The set of these assumptions will be referred to as *condition* (*).

A "non-oscillation theorem" of Hille [5, p. 245], when re-stated in the form of an "oscillation theorem", can be formulated as follows: If the pair (f, g) satisfies condition (*) and if, in addition,

$$f(t) \geqq 0, \qquad g(t) \geqq 0\,, \tag{6}$$

then, in order that (1) be oscillatory, it is sufficient that (2) be oscillatory.

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Hille's proof of this comparison theorem involves successive approximations, and the restriction (6) is essential in the resulting proof.

Actually, whereas (6) and the convergence of the integrals (3) imply that

$$(7) \qquad \int^{\infty} |f(t)|\,dt < \infty, \qquad \int^{\infty} |g(t)|\,dt < \infty\,,$$

(7) and (5) do not imply (6), and the proof is such as to fail even if (7) is assumed but (6) is omitted. It will be shown, however, that not only (7) but even (6) can be dispensed with if the proof based on the method of successive approximations is replaced by a more primitive approach. The latter will consist of an appropriate use of that simple argument on which all considerations on "disconjugation" depended in [8], where these matters were abstracted from Sturm's theorems on the one hand and from a fact occurring in the proof of Jacobi's criterion (cf., e.g., [2, pp. 57–59]) on the other hand. For other uses of that argument, cf. [3] and [4, pp. 216–217].

2. It is well known that if $f(t)$ is any continuous function satisfying

$$(8) \qquad f(t) \geqq 0 \quad \text{and} \quad \int^{\infty} f(t)\,dt = \infty\,,$$

then (1) must be oscillatory[1]; in fact, this follows by a trivial argument of convexity (cf., e.g., [6, p. 97]). But as observed in [7, p. 115], a slight refinement of that argument proves that (1) must be oscillatory even if (8) is relaxed to

$$(9) \qquad \int^{T} f(t)\,dt \to \infty \quad \text{as} \quad T \to \infty$$

[1] It is worth mentioning that *the oscillation criterion* (8) *contains a refinement of itself*, as follows: *If* $f(t)$ *is continuous and non-negative on* (3), *and if*

$$\int^{\infty} f^{\frac{1}{2}+\varepsilon}(t)\,dt = \infty$$

for some $\varepsilon > 0$, *then* (1) *is oscillatory* (this reduces to (8) if ε is chosen to be $\frac{1}{2}$). The limiting case $\varepsilon = 0$ is not allowed, since the case $\varepsilon = 0$ of the integral condition is satisfied by $f(t) = (2t)^{-2}$ but (1) has then the solution $x(t) = t^{\frac{1}{2}}$, which is non-oscillatory.

First, if $[f(t)]$ denotes the integral of f over some half-line const. $\leqq t < \infty$, then the oscillation criterion (8) can be re-stated as follows: (I) If $f(t) \geqq 0$ and if (1) is non-oscillatory, then $[f(t)] < \infty$. But as observed by P. Hartman (Amer. J. Math., vol. 74 (1952), p. 397), this criterion (I) contains, after a change of variables, the following refinement (II) of its own: (II) If $f(t) \geqq 0$ and if (1) is non-oscillatory, then $[f(t)t^{\theta}] < \infty$ holds for every fixed $\theta < 1$. On the other hand, since $[t^{-p}] < \infty$ if $p > 1$, Hölder's inequality shows that if $[f(t)t^{\theta}] < \infty$ for some positive $\theta < 1$, then $[f^{\frac{1}{2}+\varepsilon}(t)] < \infty$ for some $\varepsilon > 0$. Hence, the italicized oscillation criterion follows from the re-statement (II) of (I).

(provided that $f(t)$ is real-valued and continuous). It is precisely the possibility of generalizing (8) to (9) which suggested the possibility of removing the restriction (6) from Hille's comparison theorem. What thus results is the following criterion:

(i) *If* (f, g) *is a pair of functions satisfying condition* (*), *then* (1) *must be oscillatory whenever* (2) *is oscillatory.*

Since the assertion of (i) is equivalent to the statement that, under the (*)-assumption, (2) must be non-oscillatory whenever (1) is non-oscillatory, substantially more than the assertion of (i) is contained in the following assertion:

(ii) *If* (f, g) *is a pair of functions satisfying condition* (*), *and if* (1) *is disconjugate on a given half-line*

$$t_0 < t < \infty, \tag{10}$$

then (2) *must be disconjugate on* (10).

In fact, the disconjugate character of (1) on (10) implies that (1) has *some* (real-valued) solution $x = x(t)$ which has no zero on (10).

3. The proof of (ii) proceeds as follows:

If the assumptions of (ii) are satisfied when t_0 is fixed in (10), then they are satisfied when t_0 is replaced by any $t_0 + \varepsilon > t_0$ in (10). But then some solution $x(t)$ of (1) has no zero for $t > t_0 + \varepsilon$. Hence it will be sufficient to prove that, under this assumption, (2) is disconjugate on the half-line $t_0 + \varepsilon < t < \infty$ when $\varepsilon > 0$ is fixed. In fact, the disconjugate character of (2) on (10) will then follow by letting $\varepsilon \to 0$. Thus it is clear that it is sufficient to prove (ii) under the assumption that (1) possesses a (real-valued) solution $x(t)$ which has *no* zero on the *fixed* half-line (10).

Starting with such an $x(t)$, it is possible to form the logarithmic derivative $l = x'/x$ (Riccati) as a continuously differentiable function $l(t)$ on (10.) Suppose that the first of the integrals (3) is convergent, and define $F(t)$ by the first of the relations (4). Then $l' = -f - l^2$. It follows that

$$l(T) + \int_t^T l^2(s)\,ds \tag{11}$$

has a limit when $T \to \infty$. If the integral

$$L(t) = \int_t^\infty l^2(s)\,ds$$

diverges, it is clear that $l(T)$ tends to $-\infty$, in which case, however, the assumption of a finite upper bound for (11) readily leads to a contradiction. Thus $\lim l(T)$, where $T \to \infty$, exists and is 0 by necessity; so that $L(t)+F(t)=l(t)$. This identity can be written in the form

$$[L(t)+F(t)]^2 + L'(t) = 0 .$$

Hence, if $G(t)$ is any continuous function satisfying (5) on (10), then, since function (11) is non-negative throughout, the equation

$$p(t) + [L(t)+G(t)]^2 + L'(t) = 0$$

defines on (10) a continuous *non-negative* function $p(t)$.

4. Let the latter equation be written in the form

$$L' + L^2 + 2G(t)L + [G^2(t)+p(t)] = 0 , \tag{12}$$

and let (12) be interpreted as a differential equation for $L=L(t)$, with $G(t)$ and $p(t)$ as given coefficient functions. This non-linear differential equation has some solution which exists on the entire half-line (10); in fact, the function (11) supplies such a solution of (12). On the other hand, (12) can be interpreted as the Riccati equation belonging to the homogeneous, linear differential equation

$$u'' + 2G(t)u' + [G^2(t)+p(t)]u = 0 , \tag{13}$$

the L of (12) being Riccati's $(\log u)'=u'/u$ for (13). Since some solution of (12) exists on the whole of (10), it follows that (13) is disconjugate on (10) (this is the crucial step, the appeal to the argument of [8] on disconjugation, referred to at the end of Section 1 above).

Since (13) is disconjugate on (10), and since $p(t) \geqq 0$, it now follows from Sturm's comparison theorem that the homogeneous, linear differential equation

$$v'' + 2G(t)v' + G^2(t)v = 0 \tag{14}$$

is disconjugate on (10). Hence, the proof of (ii) will be complete if it is shown that, by virtue of the second of the relations (4), a relation which was not used thus far, (2) must be disconjugate on (10) if (14) is disconjugate on (10). But this can be verified as follows:

Choose a point $t=c$ on the half-line (10) and, in terms of any solution $u(t)$ of (13), define on (10) a function $y(t)$ by placing

$$y(t) = v(t)\exp\int_c^t G(r)\,dr, \qquad \text{where} \qquad G(t) = \int_t^\infty g(s)\,ds , \tag{15}$$

as in (4). A direct substitution shows that (14) is identical with (2) by virtue of (15). But since the exp-factor in (15) has no zero, the zeros of a solution of (2) are the same as the zeros of the corresponding solution of (12).

The proof of (ii) is now complete. Hence (i), being a weakened form of (ii), is also proved.

5. The simplest corollary is the following oscillation theorem:

(iii) *For large positive t, let $f(t)$ be a real-valued, continuous function for which the integral*

$$F(t) = \int_t^\infty f(s)\,ds, \qquad \text{where} \qquad \int_t^\infty = \lim_{T\to\infty}\int_t^T,$$

is convergent (possibly just conditionally). Then (1) *is oscillatory whenever $F(t) > C^2/t$ holds for some constant $C > \frac{1}{2}$* (and the latter condition is optimal, since $C = \frac{1}{2}$ is not allowed; cf. the case $C = \frac{1}{2}$ of (16) below).

That particular case of (iii) in which the assumption $F(t) > C^2 t$ is strengthened to $f(t) > (C/t)^2$, where $C > \frac{1}{2}$, follows from Sturm's comparison theorem, since, as observed by A. Kneser (cf. [1, p. 235]), the case

$$f(t) = (C/t)^2 \tag{16}$$

of (1) is oscillatory or non-oscillatory according as $C > \frac{1}{2}$ or $0 \leqq C \leqq \frac{1}{2}$. Hence, that particular case of (iii) in which $f(t)$ is assumed to be non-negative, follows from Hille's comparison theorem, referred to above. Correspondingly, in order to obtain (iii) in the general case, it is sufficient to apply (i) so as to compare (1) with the case (16) of (1).

6. For the same reasons, the "non-oscillatory" counterpart of the "oscillatory" criterion (iii), the counterpart of which is Satz (4.4. VI) in [1, p. 231], is a corollary of (ii). In fact, that counterpart of (iii) states that if

$$\int^\infty f^+(t)\,dt < \infty, \qquad \text{where } f^+ = \min(0, f) \tag{17}$$

(so that $f^+(t) \geqq 0$), and if

$$\int_t^\infty f^+(s)\,ds \leqq (4t)^{-1} \tag{18}$$

on (10), where $t_0 \geqq 0$, then (1) is non-oscillatory, and even disconjugate on (10). But (17) need not hold when it is only assumed that the integral

$$\int^\infty f(t)\,dt, \qquad \text{where} \qquad \int^\infty = \lim_{T\to\infty}\int^T, \tag{19}$$

is convergent; and even if (19) is *absolutely* convergent, (18) need not hold if

$$(20) \qquad 0 \leqq \int_t^\infty f(s)ds \leqq (4t)^{-1} .$$

It turns out, however, that (17) and (18) can be reduced to the (possibly just conditional) convergence of (19) and to (20). In other words, the situation is as follows:

(iv) *If* $f(t)$ *is a continuous function for which the integral* (19) *is convergent, and if* (20) *holds on a half-line* (10), *where* $t_0 \geqq 0$, *then* (1) *is non-oscillatory, and even disconjugate on* (10).

In fact, since the case (16) of (1) is disconjugate on $0 < t < \infty$ if $C = \frac{1}{2}$ (for $x(t) = t^{\frac{1}{2}}$ is then a solution), it is clear that (iv) follows by interchanging f and g in (ii).

The constant $C^2 = 4^{-1}$ occurring in the second of the inequalities (20) is the best absolute constant. In addition, the first of the inequalities (20) is essential. In fact, (1) *can be oscillatory if the integral occurring in* (20) *is allowed to become negative* (*for certain* t) *but all other assumptions of* (iv) *are satisfied.* For otherwise it would follow that (1) must be non-oscillatory whenever $f(t)$ is a continuous function satisfying

$$\int_t^\infty f(s)ds \leqq 0 \qquad \text{for} \qquad t_0 < t < \infty$$

(provided the integral is convergent). But this is readily disproved by a (piecewise) construction of an appropriate f (which is positive on *certain* t-intervals). This means that even the simplest converse of the oscillation criterion (9) is false.

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THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD., U.S.A.