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ON DIRECT DECOMPOSITIONS OF TORSIONFREE ABELIAN GROUPS

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In Kuroš [9] the question is raised whether the decomposition of a group into a direct product of finitely many indecomposable factors, when such a decomposition exists, is always essentially unique; that is, whether two such decompositions of the same group always have an equal number of factors which can be paired in such a way that corresponding factors are isomorphic. The solution to this problem is negative; in fact, in Jónsson [4] a torsionfree abelian group of finite rank is constructed, which has two essentially different decompositions into indecomposable factors.[1] Another counter-example is given in Kuroš [6], the group constructed there is finitely generated but not abelian.

The example in Jónsson [4], which is studied in greater detail in Section 1 of this note, also shows that the direct product of two torsionfree abelian groups of finite rank may have a factor of rank one although neither group has such a factor. This is particularly interesting in view of the result in Baer [1, p. 77] which states that if a group G is the direct product of finitely many torsionfree groups of rank one, then every factor of G also has this property.

In Section 2 it is shown by means of an example that the cancellation law, which is known to hold for cyclic factors (Walker [11][2]), cannot be extended to generalized cyclic factors. Section 3 contains an example of a torsionfree abelian group G of finite rank, having two decompositions $G=A\times B=C\times D$ such that A and B are isomorphic to each other, and C and D are isomorphic to each other, but A and B are not isomorphic to C and D. In terms of isomorphism types this means that the type of G has at least two distinct square roots.

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[1] It is claimed in Kuroš [7, p. 205] (see also Kuroš [8, p. 215]) that this example is incorrect, but no reason is given for this assertion. Neither the author nor others who have checked the details have been able to find an error.

[2] The fact that the cancellation law holds for *finite* factor was proved in Jónsson–Tarski [5], not only for groups, but for arbitrary algebraic systems with a zero element.

In connection with this last example we observe that the uniqueness question for square roots of isomorphism types was investigated in Hanf [2], Tarski [10] and Jónsson [3], and that the answer was shown there to be negative for various classes of algebras. However, an essential feature of the counterexamples constructed there is the fact that they are centerless, and the methods used in these papers are therefore not applicable to abelian groups.

1. The unique decomposition problem. Consider a four dimensional vector space V over the field of rational numbers, let $\{x, y, z, u\}$ be a basis for V, and let

$$x' = 3x - y \quad \text{and} \quad y' = 2x - y .$$

Then $\{x', y', z, u\}$ is also a basis for V,

$$x = x' - y' \quad \text{and} \quad y = 2x' - 3y' .$$

Let A, B, C and D consist of all those elements of V which can be written in the forms

$$\frac{a}{5^n}x, \qquad \frac{b}{5^n}y + \frac{c}{7^n}z + \frac{d}{11^n}u + \tfrac{1}{3}e(y+z) + \tfrac{1}{2}f(y+u) ,$$

$$\frac{a}{5^n}x' + \frac{c}{7^n}z + \tfrac{1}{3}e(x'-z) \quad \text{and} \quad \frac{b}{5^n}y' + \frac{d}{11^n}u + \tfrac{1}{2}f(y'-u) ,$$

respectively, with a, b, c, d, e, f and n being integers.

It is easy to see that A, B, C and D are subgroups of V under vector addition. Since A is contained in the vector space spanned by $\{x\}$ while B is contained in the vector space spanned by $\{y, z, u\}$, we see that A and B have only the zero vector 0 in common, so that their direct product $A \times B$ exists and is a subgroup of V. For similar reasons $C \times D$ exists and is a subgroup of V.

In order to show that $A \times B = C \times D$ we observe that if n is any integer, then the elements

$$\frac{1}{5^n}x = \frac{1}{5^n}x' - \frac{1}{5^n}y', \qquad \frac{1}{5^n}y = \frac{2}{5^n}x' - \frac{3}{5^n}y', \qquad \frac{1}{7^n}z, \qquad \frac{1}{11^n}u$$

belong to $C \times D$. Since the elements

$$\tfrac{1}{3}(y+z) = x' - y' - \tfrac{1}{3}(x'-z), \qquad \tfrac{1}{2}(y+u) = x' - y' - \tfrac{1}{2}(y'-u)$$

also belong to $C \times D$, we infer that $A \times B \subseteq C \times D$. On the other hand, the elements

$$\frac{1}{5^n}x' = \frac{3}{5^n}x - \frac{1}{5^n}y, \qquad \frac{1}{5^n}y' = \frac{2}{5^n}x - \frac{1}{5^n}y, \qquad \frac{1}{7^n}z, \qquad \frac{1}{11^n}u$$

belong to $A \times B$ for any integer n, and the elements

$$\tfrac{1}{3}(x'-z) = x - \tfrac{1}{3}(y+z), \qquad \tfrac{1}{2}(y'-u) = x - \tfrac{1}{2}(y+u)$$

also belong to $A \times B$, whence it follows that $C \times D \subseteq A \times B$. Thus $A \times B = C \times D$.

The group A is of rank one and is therefore indecomposable. Suppose $C = C' \times C''$. Then there exist homomorphisms φ and ψ which map C onto C' and C'', respectively, in such a way that

$$\begin{aligned} &\varphi(v)+\psi(v) = v \quad \text{for every} \quad v \in C\,, \\ &\varphi(v) = v \quad \text{and} \quad \psi(v) = 0 \quad \text{for every} \quad v \in C'\,, \\ &\varphi(v) = 0 \quad \text{and} \quad \psi(v) = v \quad \text{for every} \quad v \in C''\,. \end{aligned}$$

Observe that φ preserves rational multiples; that is, if $v \in C$ and r is a rational number such that $rv \in C$, then $r\varphi(v) \in C'$ and $\varphi(rv) = r\varphi(v)$. Since the only elements $v \in C$ with the property that $(1/5^p)v \in C$ for every integer p are the elements of the form $(a/5^n)x'$, we infer that $\varphi(x') = (a/5^n)x'$ for some integers a and n. For similar reasons $\psi(x') = (b/5^m)x'$ where b and m are integers. Hence $\psi\varphi(x') = (ab/5^{m+n})x'$. But $\psi\varphi(v) = 0$ for every $v \in C$, so that $a = 0$ or $b = 0$. We may assume that $b = 0$, in which case $\psi(x') = 0$ and $\varphi(x') = x'$. In exactly the same way we see that $\varphi(z) = 0$ or $\psi(z) = 0$. If $\varphi(z) = 0$, then

$$\varphi\big(\tfrac{1}{3}(x'-z)\big) = \tfrac{1}{3}\varphi(x') - \tfrac{1}{3}\varphi(z) = \tfrac{1}{3}x'\,,$$

which is impossible because $\frac{1}{3}x' \notin C$. We must therefore have $\psi(z) = 0$. Since every element of C is a linear combination of x' and z with rational coefficients, it follows that $\psi(v) = 0$ for every $v \in C$. We therefore conclude that $C'' = \{0\}$ and $C' = C$. Thus C is indecomposable.

A similar argument shows that D is indecomposable. It can also be shown that B is indecomposable, but for our present purpose this is not needed. For even if B were decomposable, the resulting decomposition of $A \times B$ would still contain the factor A of rank one, and since $A \times B$ is also the direct product of the two indecomposable groups C and D of rank two, it follows that $A \times B$ does not have the unique factorization property.

2. The cancellation law. Let V be a three dimensional vector space over the field of rational numbers, let $\{x, y, z\}$ be a basis for V, and let

$$x' = 8x+3y \qquad \text{and} \qquad y' = 5x+2y\,.$$

Then $\{x', y', z\}$ is also a basis for V,

$$x = 2x' - 3y' \qquad \text{and} \qquad y = -5x' + 8y' .$$

Let P and Q be two infinite disjoint sets of positive prime integers, neither of which contains the prime 5, let R be the set of all positive, squarefree integers all of whose prime factors belong to P, and let S be the set of all positive squarefree integers all of whose prime factors belong to Q. Let A, B, C and D consist of all those elements of V which can be written in the forms

$$\frac{a}{r}\,x, \qquad \frac{b}{r}\,y+\frac{c}{s}\,z+\frac{d}{5}\,(y+z)\,,$$

$$\frac{a}{r}\,x', \qquad \text{and} \qquad \frac{b}{r}\,y'+\frac{c}{s}\,z+\frac{d}{5}\,(3y'+z)\,,$$

respectively, where a, b, c and d are integers, $r \in R$ and $s \in S$. As before we see that A, B, C and D are subgroups of V and that the direct products $A \times B$ and $C \times D$ exist. It is also clear that

$$\frac{1}{r}\,x,\ \frac{1}{r}\,y,\ \frac{1}{s}\,z \ \in\ C \times D \qquad \text{and} \qquad \frac{1}{r}\,x',\ \frac{1}{r}\,y',\ \frac{1}{s}\,z \ \in\ A \times B$$

for every $r \in R$ and $s \in S$. Furthermore

$$\tfrac{1}{5}(y+z) = -x'+y'+\tfrac{1}{5}(3y'+z) \in C \times D\,,$$

$$\tfrac{1}{5}(3y'+z) = 3x+y+\tfrac{1}{5}(y+z) \in A \times B\,,$$

and we infer that $A \times B = C \times D$. Observing that A and C are isomorphic, we shall prove that B and D are not isomorphic.

Suppose there does exist a function φ which maps B isomorphically onto D. Observe that y and $-y$ are the only elements $v \in B$ with the property that, for every positive integer r, $(1/r)v \in B$ if and only if $r \in R$. Similarly y' and $-y'$ are the only elements $v \in D$ with the property that, for every positive integer r, $(1/r)v \in D$ if and only if $r \in R$. It follows that $\varphi(y) = \pm y'$, and exactly the same kind of reasoning shows that $\varphi(z) = \pm z$. Consequently

$$\varphi\big(\tfrac{1}{5}(y+z)\big) = \tfrac{1}{5}\varphi(y)+\tfrac{1}{5}\varphi(z) = \pm\tfrac{1}{5}(y' \pm z)\,,$$

which is impossible because $\tfrac{1}{5}(y' \pm z) \notin D$. We therefore conclude that no such isomorphism φ exists.

3. The square root problem. Let V be a four dimensional vector space over the field of rational numbers, let $\{x, y, z, u\}$ be a basis for V, and let

$$x' = 2x+z, \qquad y' = y+3u, \qquad z' = 17x+9z, \qquad u' = y+2u\,.$$

Then $\{x', y', z', u'\}$ is also a basis for V and

$$x = 9x'-z', \qquad y = 2y'+3u', \qquad z = -17x'+2z', \qquad u = y'-u'\,.$$

Define the sets P, Q, R and S as in the preceeding example, and let A, B, C and D consist of all those elements of V which can be written in the forms

$$\frac{a}{r}x+\frac{b}{s}y+\frac{c}{5}(x+y), \qquad \frac{a}{r}z+\frac{b}{s}u+\frac{c}{5}(z+u)\,,$$

$$\frac{a}{r}x'+\frac{b}{s}y'+\frac{c}{5}(x'+2y') \quad \text{and} \quad \frac{a}{r}z'+\frac{b}{s}u'+\frac{c}{5}(z'+2u')\,,$$

respectively, where a, b and c are integers, $r \in R$ and $s \in S$. Clearly A, B, C and D are subgroups of V and the direct products $A \times B$ and $C \times D$ exist and are subgroups of V. Furthermore, A and B are isomorphic to each other, and so are C and D. Since

$$\frac{1}{r}x,\ \frac{1}{s}y,\ \frac{1}{r}z,\ \frac{1}{s}u \ \in\ C \times D \quad \text{and} \quad \frac{1}{r}x',\ \frac{1}{s}y',\ \frac{1}{r}z',\ \frac{1}{s}u' \ \in\ A \times B$$

for every $r \in R$ and $s \in S$, and since

$$\begin{aligned}
\tfrac{1}{5}(x+y) &= 2x'+u'-\tfrac{1}{5}(x'+2y')-\tfrac{1}{5}(z+2u') \in C \times D\,,\\
\tfrac{1}{5}(z+u) &= -3x'+y'-u'-\tfrac{2}{5}(x'+2y')+\tfrac{2}{5}(z'+2u') \in C \times D\,,\\
\tfrac{1}{5}(x'+2y') &= u+\tfrac{2}{5}(x+y)+\tfrac{1}{5}(z+u) \in A \times B\,,\\
\tfrac{1}{5}(z'+2u') &= 3x+z+\tfrac{8}{5}(x+y)+\tfrac{4}{5}(z+u) \in A \times B\,,
\end{aligned}$$

we see that $A \times B = C \times D$. Finally, in order to show that B and D are not isomorphic we reason as in the second example: If φ is a function which maps B isomorphically onto D, then $\varphi(z) = \pm z'$ and $\varphi(u) = \pm u'$. Hence

$$\varphi\big(\tfrac{1}{5}(z+u)\big) = \pm\tfrac{1}{5}(z' \pm u')\,,$$

which is impossible because $\frac{1}{5}(z' \pm u') \notin D$. Thus A and B are not isomorphic to C and D.

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