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LOCAL AND GLOBAL PROPERTIES OF FUNDAMENTAL SOLUTIONS

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0. Introduction. Let $P(D)=P(i^{-1}\partial/\partial x^1, \ldots, i^{-1}\partial/\partial x^n)$ be a partial differential operator with constant coefficients. (For this and other notations used here, we also refer the reader to Hörmander [2].) By a fundamental solution of $P(D)$ we shall mean a distribution E such that $P(D)E=\delta_0$, the Dirac measure at 0 (Schwartz [5]). The fact that every partial differential operator with constant coefficients has a fundamental solution was first proved by Malgrange [3] and by Ehrenpreis [1]. More precisely, in these papers the existence of fundamental solutions of finite order and "arbitrarily small exponential growth" was established. Both of these properties are merely global. The question raised by Schwartz [5] whether there always exists a temperate fundamental solution is still open.

In his thesis [4], Malgrange proved the existence of a fundamental solution E of "small exponential growth" such that $E \star L^2{}_{\mathrm{c}} \subset L^2{}_{\mathrm{loc}}$, where $L^2{}_{\mathrm{loc}}$ ($L^2{}_{\mathrm{c}}$) is the space of locally square integrable functions (with compact support). This result, which contains the one mentioned above, means that E has a certain uniform local regularity.

The purpose of this paper is to prove that every partial differential operator $P(D)$ with constant coefficients has a fundamental solution E of "small exponential growth" (in a sense to be explained below) such that

$$Q(D)(E \star L^2{}_{\mathrm{c}}) \subset L^2{}_{\mathrm{loc}} \tag{0.1}$$

for every partial differential operator $Q(D)$ with constant coefficients which is weaker than $P(D)$ in the sense of Hörmander [2]. Such a fundamental solution we call *proper*. Conversely, it is also proved that (0.1) cannot hold for any fundamental solutions E of $P(D)$ unless Q is weaker than P.

We also give examples which prove that in general there does not exist any proper temperate fundamental solution (Section 3). This does

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not solve the problem of Schwartz mentioned above, but it shows that the global restriction of being temperate is not always compatible with the natural local properties which we require.

From the results of Hörmander [2] various properties of proper fundamental solutions can be deduced. In particular, a necessary and sufficient condition for the existence of a fundamental solution which is a locally square integrable function is that

$$\int d\xi/\tilde{P}(\xi)^2 < \infty, \tag{0.2}$$

where $\tilde{P}(\xi)^2$ is the sum of the squares of P and all its derivatives. If (0.2) holds, every proper fundamental solution is locally square integrable. — The author does not know any condition for the existence of a locally integrable fundamental solution.

Our method of constructing fundamental solutions is a modification of that of Malgrange [4], and thus depends on the Hahn-Banach theorem. It may be remarked here that the result of Malgrange can be obtained by an explicit construction. This was done by the author in an unpublished manuscript. Using this approach, Trêves [7] has recently studied fundamental solutions of differential operators depending on a parameter. However, the author has not been able to construct proper fundamental solutions in that way.

1. Construction of a fundamental solution. A fundamental solution of the differential operator $P(D)$ is by definition a distribution E such that $P(D)E=\delta_0$, the Dirac measure at 0. An equivalent definition is that $E\star\big(P(D)u\big)=u$ for all $u\in C_0^\infty$, and consequently for all distributions u with compact support. It is in fact sufficient that

$$\big(E\star\big(P(D)u\big)\big)(0) = u(0), \quad u\in C_0^\infty, \tag{1.1}$$

for applying this to all the translated functions $u(\cdot+x)$, it follows that $E\star\big(P(D)u\big)=u$. Our aim is to construct a fundamental solution E such that $Q(D)(E\star L^2{}_{\mathrm{c}})\subset L^2{}_{\mathrm{loc}}$ for as many differential operators $Q(D)$ as possible.

THEOREM 1.1. *Suppose that $P(D)$ has a fundamental solution E such that*

$$Q(D)(E\star L^2{}_{\mathrm{c}}) \subset L^2{}_{\mathrm{loc}}. \tag{1.2}$$

Then Q must be weaker than P, that is,

$$\tilde{Q}(\xi)/\tilde{P}(\xi) < C \quad \textit{when } \xi \textit{ is real}, \tag{1.3}$$

where

$$\tilde{P}(\xi) = (\Sigma |P^{(\alpha)}(\xi)|^2)^{\frac{1}{2}}, \quad \tilde{Q}(\xi) = (\Sigma |Q^{(\alpha)}(\xi)|^2)^{\frac{1}{2}}, \tag{1.4}$$

the sums being extended over the derivatives of all orders $\geqq 0$ *of* P *and* Q, *respectively.*

PROOF. Suppose that (1.2) is valid. Let u and $P(D)u$ be in $L^2{}_c$. Since u has compact support it follows that $u = E \star (P(D)u)$, and hence

$$Q(D)u = Q(D)\big(E \star (P(D)u)\big) \in L^2{}_{loc}$$

in virtue of (1.2). Since $Q(D)u$ has compact support, we have proved that $Q(D)u \in L^2{}_c$ if u and $P(D)u$ are in $L^2{}_c$. But then it follows from Theorem 2.2 and a remark on p. 170 in Hörmander [2] that (1.3) must hold.

DEFINITION. *A fundamental solution* E *of* $P(D)$ *will be called proper if* (1.2) *is valid for every* Q *weaker than* P.

In terms of this definition, our main result is:

THEOREM 1.2. *Every differential operator with constant coefficients has a proper fundamental solution* E *of small exponential growth in the sense that* $E/\cosh\big(\varepsilon(x^2+1)^{\frac{1}{2}}\big)$ *is temperate,* ε *being any preassigned positive number.*

For the definition of a temperate distribution we refer to Schwartz [5, t. II].

In the proof of Theorem 1.2 we shall use some normed spaces instead of $L^2{}_c$ and $L^2{}_{loc}$ in order to simplify the arguments, and to obtain the second half of the statement. By $L^2{}_r$ where r is a real number we shall denote the Hilbert space of all functions u such that $u(x)e^{r|x|}$ is square integrable. The norm in $L^2{}_r$ is defined by

$$\|u\|_r{}^2 = \int |u(x)|^2 e^{2r|x|}\,dx\,. \tag{1.5}$$

Let ε be an arbitrary but fixed positive number. We shall prove the existence of a fundamental solution E such that the mapping

$$u \to Q(D)(E \star u)$$

can be extended from C_0^∞ to a continuous mapping from $L^2{}_\varepsilon$ to $L^2{}_{-\varepsilon}$. As we shall see later on, this contains Theorem 1.2.

In the proof of Theorem 1.2 we also need another norm, namely

$$N(u) = \sup_{|\eta| \leqq \varepsilon} \int |\hat{u}(\xi + i\eta)|/\tilde{P}(\xi)\, d\xi, \quad u \in C_0^\infty, \tag{1.6}$$

where $\hat{u}$ is the Fourier Laplace transform of u,

$$\hat{u}(\zeta) = \int e^{-i\langle x, \zeta\rangle} u(x)\, dx\, .$$

THEOREM 1.3. *There exists a fundamental solution E of $P(D)$ such that*

$$|E \star u(0)| \leqq CN(u), \quad u \in C_0^\infty\, . \tag{1.7}$$

Before proving this theorem we shall prove that Theorem 1.2 follows from Theorem 1.3. Thus let E be a fundamental solution of $P(D)$ for which (1.7) is valid. Replace u by $Q(D)u \star v$, where u and v are in C_0^∞ and Q is weaker than P. This gives

$$|Q(D)E \star u \star v(0)| \leqq CN\big(Q(D)u \star v\big)\, . \tag{1.8}$$

In order to estimate the right-hand side of this inequality we note that the Fourier-Laplace transform of $Q(D)u \star v$ is $Q(\zeta)\hat{u}(\zeta)\hat{v}(\zeta)$. Since

$$Q(\xi + i\eta) = \Sigma\, Q^{(\alpha)}(\xi)(i\eta)_\alpha/|\alpha|!$$

and Q is weaker than P, we have

$$|Q(\xi + i\eta)|/\tilde{P}(\xi) \leqq C'$$

when $|\eta| \leqq \varepsilon$ and ξ is real. Hence

$$N\big(Q(D)u \star v\big) \leqq C' \sup_{|\eta| \leqq \varepsilon} \int |\hat{u}(\xi + i\eta)\, \hat{v}(\xi + i\eta)|\, d\xi\, . \tag{1.9}$$

Using Parseval's equality and $\langle x, \eta\rangle \leqq |x|\,|\eta|$ we obtain

$$(2\pi)^{-n} \int |\hat{u}(\xi + i\eta)|^2\, d\xi = \int |u(x)|^2 e^{2\langle x, \eta\rangle} dx \leqq \|u\|^2_\varepsilon, \quad |\eta| \leqq \varepsilon\, ,$$

and a similar inequality for $\hat{v}$, so that with Cauchy-Schwarz' inequality applied to the right-hand side of (1.9) we get

$$N\big(Q(D)u \star v\big) \leqq C''\, \|u\|_\varepsilon \|v\|_\varepsilon, \quad u, v \in C_0^\infty\, .$$

Combination of this estimate with (1.8) gives

$$\left| \int \big(Q(D)E \star u\big)(x)\, v(-x)\, dx \right| \leqq CC''\, \|u\|_\varepsilon \|v\|_\varepsilon, \quad u, v \in C_0^\infty\, . \tag{1.10}$$

Since C_0^∞ is dense in $L^2{}_\varepsilon$, and $L^2{}_{-\varepsilon}$ is the dual space of $L^2{}_\varepsilon$, this gives if we divide by $\|v\|_\varepsilon$ and take the supremum over v

$$\|Q(D)E \star u\|_{-\varepsilon} \leqq CC''\, \|u\|_\varepsilon, \quad u \in C_0^\infty\, . \tag{1.11}$$

Hence the mapping $u \to Q(D)E \star u$ can be extended (by continuity) from C_0^∞ to $L^2{}_\varepsilon$, so that it becomes a continuous mapping from $L^2{}_\varepsilon$ to $L^2{}_{-\varepsilon}$. For $u \in L^2{}_c$ it is obvious that $Q(D)E \star u$ is the same with this definition as with that of Schwartz [5]. In particular we get $Q(D)E \star L^2{}_c \subset L^2{}_{-\varepsilon} \subset L^2{}_{\mathrm{loc}}$.

Also note that it follows from (1.7) that

$$E \star u_n(0) \to 0$$

if $u_n(x) \cosh\bigl(\varepsilon(x^2+1)^{\frac{1}{2}}\bigr) \to 0$ in $\mathscr{S}$ (cf. Schwartz [5] for this notation). For then we have that

$$\int |\hat{u}_n(\xi + i\eta)|\, d\xi \to 0$$

uniformly in η when $|\eta| \leqq \varepsilon$ so that $N(u_n) \to 0$. But this means that $E / \cosh\bigl(\varepsilon(x^2+1)^{\frac{1}{2}}\bigr)$ is continuous on $\mathscr{S}$, hence is a temperate distribution. Thus we have proved that Theorem 1.3 implies Theorem 1.2.

Proof of Theorem 1.3. We have to construct a linear form $L(u) = E \star u(0)$, defined in C_0^∞, such that (1.7) holds, that is,

$$|L(u)| \leqq CN(u), \quad u \in C_0^\infty,$$

and the definition of a fundamental solution is satisfied, or

$$L\bigl(P(D)u\bigr) = u(0), \quad u \in C_0^\infty.$$

Hahn-Banach's theorem shows that a linear form with these properties exists if and only if

$$|u(0)| \leqq CN\bigl(P(D)u\bigr), \quad u \in C_0^\infty. \tag{1.12}$$

We shall prove this inequality by means of a slight extension of the arguments of Malgrange [4], and will then have accomplished a proof of Theorems 1.2 and 1.3.

Lemma 1.1. (Malgrange.) *If $f(t)$ is an analytic function of a complex variable t when $|t| \leqq 1$ and $p(t)$ is a polynomial in which the coefficient of the highest order term is A, then*

$$|Af(0)| \leqq (2\pi)^{-1} \int_{-\pi}^{\pi} |f(e^{i\varphi})p(e^{i\varphi})|\, d\varphi. \tag{1.13}$$

Proof. Let m be the degree of p, and q be the polynomial $q(t) = t^m \bar{p}(1/t)$, where $\bar{p}$ is obtained by conjugating the coefficients of p. We then have $q(0) = \bar{A}$ and $|q(e^{i\varphi})| = |p(e^{i\varphi})|$ so that (1.13) reduces to the well-known inequality

$$|f(0)q(0)| \leqq (2\pi)^{-1} \int_{-\pi}^{\pi} |f(e^{i\varphi})q(e^{i\varphi})|\, d\varphi.$$

Lemma 1.2. *With the notations of Lemma* 1 *we have, if the degree of* p *is* $\leqq m$,

$$|f(0)p^{(k)}(0)| \leqq \frac{m!}{(m-k)!}(2\pi)^{-1}\int_{-\pi}^{\pi}|f(e^{i\varphi})p(e^{i\varphi})|\,d\varphi\,. \tag{1.14}$$

Proof. We may assume that the degree of p is m and that

$$p(t) = \prod_1^m (t-t_j)\,.$$

Applying the previous lemma to the polynomial $\Pi_1^k(t-t_j)$ and the analytic function $f(t)\,\Pi_{k+1}^m(t-t_j)$, we obtain

$$|f(0)|\left|\prod_{k+1}^m t_j\right| \leqq (2\pi)^{-1}\int_{-\pi}^{\pi}|f(e^{i\varphi})p(e^{i\varphi})|\,d\varphi\,.$$

A similar inequality will hold for any product of $m-k$ of the numbers t_j on the left-hand side, and since $p^{(k)}(0)$ is the sum of $m!/(m-k)!$ such terms, multiplied by $(-1)^{m-k}$, the inequality (1.14) follows.

Note that (1.14) reduces to (1.13) when $k=m$ and is trivial when $k=0$.

We shall now rewrite Lemma 1.2 in a form which facilitates the extension to several variables. Suppose, for simplicity in the statement, that f is entire and apply (1.14) to the functions $f(rt)$ and $p(rt)$, where $r>0$. This gives

$$|f(0)p^{(k)}(0)|\,r^k \leqq \frac{m!}{(m-k)!}(2\pi)^{-1}\int_{-\pi}^{\pi}|f(re^{i\varphi})p(re^{i\varphi})|\,d\varphi\,.$$

Let $\psi(r)$ be a non negative integrable function with compact support. Multiplying by $2\pi r\psi(r)$ and integrating with respect to r, we obtain

$$|f(0)p^{(k)}(0)|\int|t^k|\,\psi(|t|)\,dt \leqq \frac{m!}{(m-k)!}\int|f(t)p(t)|\,\psi(|t|)\,dt\,, \tag{1.15}$$

where dt stands for the Lebesgue measure $r\,dr\,d\varphi$ and the integrals are extended over the whole complex plane. The following generalization to several variables follows immediately by applying (1.15) successively to the variables $\zeta_1, \ldots, \zeta_n$, one at a time.

Lemma 1.3. *Let* $F(\zeta)$ *be an entire analytic function and* $P(\zeta)$ *a polynomial of degree* $\leqq m$ *in* $\zeta=(\zeta_1, \ldots, \zeta_n)$. *Let* $\Psi(\zeta)$ *be a non negative integrable function with compact support depending only on* $|\zeta_1|, \ldots, |\zeta_n|$.

Then

$$(1.16)\quad |F(0)P^{(\alpha)}(0)| \int |\zeta_\alpha|\, \Psi(\zeta)\, d\zeta \leq \frac{m!}{(m-|\alpha|)!} \int |F(\zeta)P(\zeta)|\, \Psi(\zeta)\, d\zeta ,$$

where $d\zeta$ is the Lebesgue measure in C_n.

We can now prove (1.12). Let u be in C_0^∞ and write $P(D)u=v$; then $P(\zeta)\hat{u}(\zeta)=\hat{v}(\zeta)$. Apply Lemma 1.3 with $F(\zeta)=\hat{u}(\xi+\zeta)$, with $P(\zeta)$ replaced by $P(\xi+\zeta)$ and with $\Psi(\zeta)=1$ when $|\zeta|\leqq\varepsilon$ and $=0$ otherwise. Adding over all α and noting that $\tilde{P}(\xi)\leqq\Sigma|P^{(\alpha)}(\xi)|$, we obtain

$$(1.17)\quad |\hat{u}(\xi)|\tilde{P}(\xi) \leqq C_1 \int_{|\zeta|\leqq\varepsilon} |\hat{u}(\xi+\zeta)P(\xi+\zeta)|\, d\zeta = C_1 \int_{|\zeta|\leqq\varepsilon} |\hat{v}(\xi+\zeta)|\, d\zeta .$$

Hence

$$\begin{aligned} |u(0)| &= \left| (2\pi)^{-n} \int \hat{u}(\xi)\, d\xi \right| \\ &\leqq C_1' \iint_{|\zeta|\leqq\varepsilon} |\hat{v}(\xi+\zeta)| / \tilde{P}(\xi)\, d\xi d\zeta \\ &= C_1' \iiint_{|\xi'|^2+|\eta'|^2\leqq\varepsilon^2} |\hat{v}(\xi+\xi'+i\eta')| / \tilde{P}(\xi)\, d\xi d\xi' d\eta' . \end{aligned}$$

Now we have

$$(1.18)\qquad \tilde{P}(\xi+\xi')/\tilde{P}(\xi) \leqq C_2 \quad \text{when } |\xi'| \leqq \varepsilon .$$

For $P^{(\alpha)}(\xi+\xi')=\Sigma P^{(\alpha+\beta)}(\xi)\xi_\beta'/|\beta|!$ so that $|P^{(\alpha)}(\xi+\xi')|/\tilde{P}(\xi)$ is bounded when $|\xi'|\leqq\varepsilon$. This gives $\tilde{P}(\xi)/\tilde{P}(\xi+\xi')\leqq C_2$ and hence

$$\begin{aligned} |u(0)| &\leqq C_1' C_2 \iiint |v(\xi+\xi'+i\eta')| / \tilde{P}(\xi+\xi')\, d\xi d\xi' d\eta' \\ &\leqq C_1' C_2 N(v) \iint_{|\xi'|^2+|\eta'|^2\leqq\varepsilon^2} d\xi' d\eta' \\ &= CN\big(P(D)u\big) . \end{aligned}$$

The proof is complete.

We next examine the question whether all fundamental solutions may be proper.

THEOREM 1.4. *Every fundamental solution of $P(D)$ is proper if and only if $P(D)$ is complete and of local type. Every fundamental solution, such that $E\star L^2{}_c\subset L^2{}_{loc}$, is proper if and only if $P(D)$ is of local type.*

The terms "complete" and "of local type" are defined in Hörmander [2, pp. 200 and 218].

Proof. First assume that $P(D)$ is *complete and of local type*. If E_1 and E_0 are fundamental solutions, $u=E_1-E_0$ satisfies $P(D)u=0$ and is therefore an infinitely differentiable function. If we take E_0 proper, it follows that E_1 is also proper. Hence all fundamental solutions are proper in this case. (This was also pointed out in Hörmander [2, p. 223].)

Next assume that $P(D)$ is *of local type* and that $E\star L^2{}_c\subset L^2{}_{loc}$. If $f\in L^2{}_c$ and we put $u=E\star f$ we have $u\in L^2{}_{loc}$ and $P(D)u=f\in L^2{}_c$. Hence in virtue of the definition of operators of local type and Theorem 2.2 in Hörmander [2], it follows that $Q(D)u\in L^2{}_{loc}$ for all Q weaker than P. Hence E is proper.

We shall now prove the other half of the theorem. Assume that *all fundamental solutions E of $P(D)$ for which $E\star L^2{}_c\subset L^2{}_{loc}$ are proper.* Let U be the set of all solutions of $P(D)u=0$ such that $u\star L^2{}_c\subset L^2{}_{loc}$. Let E_0 be a fixed proper fundamental solution of $P(D)$. By assumption, if $u\in U$, the distribution $E=E_0+u$ is a proper fundamental solution of $P(D)$, for $E\star L^2{}_c\subset L^2{}_{loc}$. Thus $Q(D)E\star L^2{}_c\subset L^2{}_{loc}$ if $Q(D)$ is weaker than $P(D)$, and since the same is true of E_0 it follows that $Q(D)u\star L^2{}_c\subset L^2{}_{loc}$, hence $Q(D)u\in U$. — Also note that every locally square integrable solution of $P(D)u=0$ is in U.

Without restriction we may assume that the coordinate system is so chosen that $P(\xi)$ is a complete polynomial in $\xi_1, \ldots, \xi_\nu$ and is independent of $\xi_{\nu+1}, \ldots, \xi_n$. The algebra generated by the polynomials weaker than P then consists of all polynomials in $\xi_1, \ldots, \xi_\nu$ (Hörmander [2, p. 208]). Now let U_1 be the set of those solutions of $Pu=0$ which are in U and only depend on $x^1, \ldots, x^\nu$. Since $Q(D)U_1\subset U_1$ if Q is weaker than P, it follows that U_1 is invariant for differentiation with respect to $x^1, \ldots, x^\nu$ and hence for all differentiations. If $f\in L^2{}_c$ and $u\in U_1$ it thus follows from the definition of U that all derivatives of $u\star f$ are in $L^2{}_{loc}$. This proves that $u\star f$ is infinitely differentiable (Sobolev's lemma). Hence it easily follows that $D^\alpha u\in L^2{}_{loc}$, the dual space of $L^2{}_c$, for all α, and a second application of Sobolev's lemma thus shows that u is an infinitely differentiable function, if $u\in U_1$.

Now an easy modification of the proof of Theorem 3.7 in Hörmander [2] shows that, if every locally square integrable solution of a partial differential operator with constant coefficients is infinitely differentiable, then the operator is complete and of local type. In fact, one only has to consider the Hilbert space of all solutions which are square integrable with respect to $e^{-x^2}dx$ and argue in the same way as there. Hence it follows that $P(D)$ is of local type.

It remains to prove that $P(D)$ must be complete if *all fundamental solutions are proper*. Assume that $P(D)$ is not complete; we may assume that $P(D)$ does not contain any differentiation with respect to x^1. Let E_0 be a proper fundamental solution, and w an exponential solution of $P(D)$, and set

$$E(v) = E_0(v) + \int_{x'=0} \partial v/\partial x^1 \, w \, dx^2 \ldots dx^n .$$

It is immediately verified that E is a fundamental solution and that $E \star L^2{}_{\mathrm{c}}$ is not contained in $L^2{}_{\mathrm{loc}}$. Hence E is not proper.

2. The character of proper fundamental solutions. We shall here examine more carefully the properties of the convolution $E \star f$ when $f \in L^2{}_{\mathrm{c}}$ and E is a proper fundamental solution of $P(D)$.

DEFINITION. *A linear subspace V of the space $\mathscr{D}'$ of distributions is called local if*

1° *$\varphi \in C_0^\infty$ and $f \in V$ implies $\varphi f \in V$.*
2° *Every distribution f such that $\varphi f \in V$ for all $\varphi \in C_0^\infty$ is in V.*

THEOREM 2.1. *Let $P(D)$ and $Q(D)$ be two differential operators with constant coefficients and E a fundamental solution of $P(D)$. If*

$$Q(D) E \star L^2{}_{\mathrm{c}} \subset V , \tag{2.1}$$

where V is a space of distributions, it follows that

$$Q(D)u \in V \quad \textit{if} \quad u \in L^2{}_{\mathrm{c}} \quad \textit{and} \quad P(D)u \in L^2{}_{\mathrm{c}} . \tag{2.2}$$

On the other hand, if V is a local *space of distributions and* (2.2) *is true, we have* (2.1) *for every* proper *fundamental solution E.*

Thus the proper fundamental solutions are in a certain sense the best possible.

PROOF. Assume that (2.1) holds and let $u \in L^2{}_{\mathrm{c}}$, $P(D)u = v \in L^2{}_{\mathrm{c}}$. Then $u = E \star \big(P(D)u\big) = E \star v$ and hence

$$Q(D)u = Q(D) E \star v \in V .$$

This proves (2.2). Now assume that (2.2) is true and that V is a local space of distributions. Let E be a proper fundamental solution. Take $f \in L^2{}_{\mathrm{c}}$ and set $g = E \star f$. Then $P^{(\alpha)} g \in L^2{}_{\mathrm{loc}}$ for every α since E is proper. Hence if $\varphi \in C_0^\infty$ and we set $h = \varphi g$, it follows that $h \in L^2{}_{\mathrm{c}}$ and that $P(D)h = \Sigma P^{(\alpha)}(D) g D_\alpha \varphi / |\alpha| ! \in L^2{}_{\mathrm{c}}$. Thus it follows from (2.2) that $Q(D)h = Q(D)(\varphi g) \in V$. Let ψ be another function in C_0^∞, arbitrarily chosen. We may assume that $\varphi = 1$ in a neighbourhood of the support of ψ. Then we have, in a neighbourhood of this support, $Q(D)(\varphi g) = Q(D)g$ so that $\psi Q(D)g = \psi Q(D)(\varphi g) \in V$ in virtue of the definition above, 1°. Since ψ is arbitrary it follows from 2° that $Q(D)g \in V$, which proves the theorem.

Theorems 2.6, 2.7, 2.15 and others in Hörmander [2] give results of the form (2.2). Combining them with Theorem 2.1 we obtain the following three theorems, if in the first one we also note that $Q(D)E \star L^2{}_c$ is contained in the space of continuous functions if and only if $Q(D)E \in L^2{}_{loc}$.

THEOREM 2.2. *If E is a proper fundamental solution of $P(D)$, we have $Q(D)E \in L^2{}_{loc}$ if and only if*

$$\int \tilde{Q}(\xi)^2/\tilde{P}(\xi)^2\, d\xi < \infty . \tag{2.3}$$

This condition is also necessary for the existence of any fundamental solution such that $Q(D)E \in L^2{}_{loc}$.

THEOREM 2.3. *If E is a proper fundamental solution of $P(D)$, we have $Q(D)E \star L^2{}_c \subset L^p{}_{loc}$ if*

$$\tilde{Q}(\xi)/\tilde{P}(\xi) \in L^{2p/(p-2)} \quad (p > 2) . \tag{2.4}$$

THEOREM 2.4. *If E is a proper fundamental solution of $P(D)$, the mapping $u \to Q(D)E \star u$ maps bounded sets in $L^2{}_c$ into compact sets in $L^2{}_{loc}$ if and only if $\tilde{Q}(\xi)/\tilde{P}(\xi) \to 0$ when $\xi \to \infty$.*

3. The non existence of temperate proper fundamental solutions. Let $P(D)$ be a differential operator such that $P(\xi) \neq 0$ when ξ is real. If E is a temperate fundamental solution with Fourier transform $\hat{E}$, we must have $P(\xi)\hat{E} = 1$, so that $\hat{E}$ is the function $1/P(\xi)$. Since it follows from algebraic results (Seidenberg [6]) that $1/|P(\xi)| \leqq C(1+\xi^2)^m$ for suitable C and m, there exists one and only one temperate distribution with $\hat{E} = 1/P(\xi)$. We are going to give one necessary and one sufficient condition for this unique temperate fundamental solution to be proper.

THEOREM 3.1. *Suppose that $P(\xi) > 0$ and that the temperate fundamental solution E of $P(D)$ is proper. Then, if Q is weaker than P and $Q(\xi) \geqq 0$, the integral*

$$\int_{|\xi-\eta| \leqq 1} Q(\xi)/P(\xi)\, d\xi \tag{3.1}$$

must be a bounded function of η.

PROOF. When u and v have their supports in a fixed compact set K, we shall have

$$\left| \int (Q(D)E \star u)\, \bar{v}\, dx \right| \leqq C \|u\| \, \|v\| ,$$

where the norms on the right are L^2-norms. We choose a fixed function u_0 such that $|\hat{u}_0(\xi)| \geqq 1$ when $|\xi| \leqq 1$ and apply the inequality to the functions $u = v = u_0 e^{i\langle x, \eta \rangle}$ with real η. This gives

$$\int |\hat{u}_0(\xi - \eta)|^2 Q(\xi) / P(\xi)\, d\xi \leqq C' .$$

Hence the integral in (3.1) is also bounded by C'.

THEOREM 3.2. *Let $P(D)$ be a differential operator such that*

$$\int_{|\xi - \eta| \leqq 1} |P^{(\alpha)}(\xi) / P(\xi)|\, d\xi \tag{3.2}$$

is a bounded function of η for every α. In particular, the function $1/P(\xi)$ is thus supposed locally integrable and temperate so that there is a temperate distribution E with Fourier transform $1/P(\xi)$. This is a proper fundamental solution.

PROOF. It is obvious that E is a fundamental solution. To prove that E is proper we shall prove that (1.7) holds. Let u be in C_0^∞. Then

$$\begin{aligned} |E \star u(0)| &\leqq (2\pi)^{-n} \int |\hat{u}(\xi)| / |P(\xi)|\, d\xi \\ &= C \iint_{|\xi'| \leqq 1} |\hat{u}(\xi + \xi') / P(\xi + \xi')|\, d\xi\, d\xi' . \end{aligned}$$

Since by assumption the integral

$$\int_{|\xi'| \leqq 1} \tilde{P}(\xi + \xi') / |P(\xi + \xi')|\, d\xi'$$

is bounded, we obtain

$$|E \star u(0)| \leqq C' \int \Big(\sup_{|\xi'| \leqq 1} |\hat{u}(\xi + \xi')| / \tilde{P}(\xi + \xi') \Big)\, d\xi .$$

Let G be a bounded domain in the complex n-space having the real unit sphere in its interior and which is contained in the strip $|\operatorname{Im} \zeta| < \varepsilon$. Then

$$\sup_{|\xi'| \leqq 1} |F(\xi')| \leqq C'' \int_G |F(\zeta)|\, d\zeta$$

for all $F(\zeta)$ which are analytic in G, and hence, $\hat{u}$ being analytic,

$$|\hat{u}(\xi+\xi')| \leqq C'' \int_G |\hat{u}(\xi+\zeta)|d\zeta, \quad |\xi'| \leqq 1 .$$

Using this estimate and (1.18) we now obtain

$$|E\star u(0)| \leqq C''' \iiint_{\xi'+i\eta' \in G} |\hat{u}(\xi+\xi'+i\eta')| / \tilde{P}(\xi+\xi')\, d\xi\, d\xi'\, d\eta' \leqq C N(u) ,$$

which proves the theorem.

We are now going to apply these two theorems to two examples.

EXAMPLE 1. Let $P(\xi)=\Sigma_{i\neq j}\xi_i^2\xi_j^2+1$. Then the temperate fundamental solution is proper if and only if the dimension n is $\geqq 4$.

PROOF. Assume that the fundamental solution is proper. Then, by Theorem 3.1, we must have

$$\int_{|\xi-\eta|\leqq 1} \xi_1^2/P(\xi)\, d\xi \leqq C .$$

Now note that $P(\xi)=\xi_1^2(\xi_2^2+\ldots+\xi_n^2)+$ terms independent of ξ_1. Thus, if we take $\eta=(t, 0, \ldots, 0)$ and let $t\to\infty$, we obtain

$$\int_{|\xi|\leqq 1} (\xi_2^2+\ldots+\xi_n^2)^{-1}\, d\xi \leqq C .$$

Hence $n-1>2$, that is, $n\geqq 4$.

Next assume that $n\geqq 4$. Since $P(\xi)\geqq \xi_1^2(\xi_2^2+\ldots+\xi_n^2)$ it follows that

$$\int_{|\xi-\eta|\leqq 1} |Q(\xi)/P(\xi)|\, d\xi \tag{3.3}$$

is bounded when $Q(\xi)=\xi_1^2$ and hence when $Q(\xi)=\xi_i^2$. Obviously this is also true for $Q(\xi)=1$ and $Q(\xi)=\xi_i^2\xi_k^2$ with $i\neq k$ and therefore for the geometric means $Q(\xi)=\xi_i$ and $Q(\xi)=\xi_i^2\xi_k$ or $\xi_i\xi_k$ with $i\neq k$. Since all $P^{(\alpha)}$ are sums of such polynomials, it follows from Theorem 3.2 that E is proper.

EXAMPLE 2. Let $P(\xi)=(\xi_1\xi_2-1)^{2k}+\xi_1^{2k}$ with $k>1$ (two variables). We shall prove that not even $E\star L^2_{\mathrm{c}}\subset L^2_{\mathrm{loc}}$ for the temperate fundamental solution E. Examination of the proof of Theorem 3.1 shows that we have only to prove that the integral (3.1) with $Q=1$ is not bounded.

To do so we consider the integral $\int d\xi/P(\xi)$ over the set $a\leqq\xi_2\leqq a+1$, $0\leqq\xi_1\leqq 1$, with large positive a. In this set we have $P(\xi)\leqq 2/a^{2k}$ provided that $0 \leqq \xi_1 \leqq 1/a$ and $-1/a \leqq \xi_1 \xi_2 - 1 \leqq 1/a$. The latter inequality is fulfilled for all ξ_2 in the interval $(a, a+1)$ if

$$\xi_1(a+1) - 1 \leqq 1/a, \quad \xi_1 a - 1 \geqq -1/a\,,$$

that is, if $1/a - 1/a^2 \leqq \xi_1 \leqq 1/a$. This interval has length $1/a^2$ and hence the integral of $1/P(\xi)$ over the square is larger than $a^{2k}/2a^2$, which tends to infinity with a if $k > 1$.

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