AN INEQUALITY FOR FINITE SEQUENCES

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1. Let \( (a_v) \) and \( (b_v) \), \( v = 0, 1, 2, \ldots, m \), be two finite sequences of real and non-negative numbers. In this note we derive an inequality of Hilbert-Schwarz's type and, in the case of \( a_v = b_v \), we get incidentally a refined form for the well-known Hilbert's inequality

\[
\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u a_v}{u + v + 1} \leq \frac{\pi}{2} \sum_{u=0}^{m} a_u^2
\]

for the finite sequence \( (a_v) \). (Concerning this inequality and its generalisations cf. e.g. [2, pp. 117, 290] and [1, Chapt. IX].) Our result is as follows:

**Theorem I.** Let \( (a_v) \) and \( (b_v) \) be the given sequences, then

\[
\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} \leq (m + 1) \sin \frac{\pi}{2(m+1)} \left( \frac{1}{\sum_{u=0}^{m} a_u^2} \right)^{\frac{1}{2}} \left( \frac{1}{\sum_{u=0}^{m} b_u^2} \right)^{\frac{1}{2}}.
\]

In the proof we require the following almost evident relation:

**Lemma.** Let \( q, u \) and \( v \) be positive integers. If \( q \mid u - v \), then

\[
\sum_{r=0}^{n-1} e^{2i(\pi(q-1)r/n)} = \begin{cases} 
0 & \text{if } u \neq v, \\
n & \text{if } u = v.
\end{cases}
\]

2. We construct a regular polygon \( C \) with an even number \( n > 2m \) of sides, inscribed in the unit circle \( |z| = 1 \) and with the vertices \( P_r = e^{2i\pi r/n} \), where \( P_n = P_0 \). Let \( L_r, 0 \leq r \leq n - 1 \), denote the chord joining the points \( P_r \) and \( P_{r+1} \). On the chord \( L_r \) we have

\[
z = \frac{\cos \pi/n \ e^{i\theta}}{\cos((2r+1)\pi/n - \theta)}, \quad 2r\pi/n \leq \theta \leq (2r+2)\pi/n,
\]

hence

\[
|dz| = \frac{\cos \pi/n \ d\theta}{\cos^2((2r+1)\pi/n - \theta)}
\]

if \( \theta \) is measured in the positive sense of the angle.
Let $C_1$ and $C_2$ be the parts of $C$ above and below the real axis, respectively. We define two functions:

\[ f(z) = \sum_{u=0}^{m} a_u z^{2u}, \quad g(z) = \sum_{v=0}^{m} b_v z^{2v}. \]

Then we have

\[ \sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} = \frac{1}{2} \int_{C_1} f(x) g(x) \, dx = \frac{1}{2i} \int_{C_1} f(z) g(z) \, dz \]

by Cauchy’s theorem for contour integration. Let the last integral be denoted by $I$, then

\[ I = \frac{1}{2i} \int_{C_1} f(z) g(z) \, dz \leq \frac{1}{2} \int_{C} |f(z)||g(z)| \, |dz| \]

\[ \leq \frac{1}{2} \left( \int_{C} |f(z)|^2 \, |dz| \right)^{\frac{1}{2}} \left( \int_{C} |g(z)|^2 \, |dz| \right)^{\frac{1}{2}} \]

\[ = \frac{1}{2} I_1^{\frac{1}{2}} I_2^{\frac{1}{2}}, \]

say, by Schwarz’s inequality. Now

\[ I_1 = \frac{1}{2} \int_{C} f(z) f(\bar{z}) \, |dz| \]

\[ = \sum_{r=0}^{n-1} \int_{L_r} f(z) f(\bar{z}) \, |dz| \]

\[ = \sum_{r=0}^{n-1} \left( \sum_{u=0}^{m} a_u z^{2u} \right) \left( \sum_{v=0}^{m} a_v \bar{z}^{2v} \right) \, |dz| \]

\[ \sum_{r=0}^{n-1} \frac{\sum_{u=0}^{m} a_u \cos^{2u} \frac{\pi}{n} e^{2iu\theta}}{\cos^{2u} \left( (2r+1)\frac{\pi}{n} - \theta \right)} \sum_{v=0}^{m} a_v \frac{\cos^{2v} \frac{\pi}{n} e^{-2iv\theta}}{\cos^{2v} \left( (2r+1)\frac{\pi}{n} - \theta \right)} \cdot \cos \frac{\pi}{n} \, d\theta \]

\[ = \sum_{r=0}^{n-1} \left( \sum_{u=0}^{m} a_u \cos \frac{\pi}{n} \right) \left( \sum_{v=0}^{m} a_v \cos \frac{\pi}{n} \right) \frac{\cos \frac{2\pi}{n} \phi}{\cos^{2(u+v)+2} \phi} \sum_{r=0}^{n-1} e^{4i(u-v)\frac{\pi r}{n}} \, d\phi \]

\[ = n \sum_{u=0}^{m} a_u \frac{\cos^{4u+1} \phi}{\cos^{4u+2} \phi} \, d\phi \]

by the lemma of section 1. Since
\[
\int_{-\pi/n}^{\pi/n} \frac{d\varphi}{\cos^{4u+2} \varphi} \leq \frac{1}{\cos^{4u} \pi/n} \int_{-\pi/n}^{\pi/n} \frac{d\varphi}{\cos^{2} \varphi} = 2 \frac{\sin \pi/n}{\cos^{4u+1} \pi/n}
\]

for \(0 \leq u \leq m\), we obtain immediately that

\[
I_1 \leq 2n \sin \pi/n \sum_{u=0}^{m} a_u^2.
\]

A similar argument gives

\[
I_2 \leq 2n \sin \pi/n \sum_{u=0}^{m} b_u^2.
\]

From the above analysis it follows that

\[
\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} \leq \frac{1}{n} \sin \pi/n \left( \sum_{u=0}^{m} a_u^2 \right)^{1/2} \left( \sum_{u=0}^{m} b_u^2 \right)^{1/2}
\]

for \(n > 2m\), \(n\) even. Choosing \(n = 2(m+1)\), we obtain theorem 1.

3. We can easily remove the restriction to non-negative \(a_u\) and \(b_v\), by observing that

\[
\left| \sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} \right| \leq \sum_{u=0}^{m} \sum_{v=0}^{m} \frac{|a_u| |b_v|}{2u + 2v + 1}.
\]

Thus we obtain

**Theorem 2.** Let \((a_u)\) and \((b_v)\), \(u = 0, 1, \ldots, m\), be two finite sequences of real numbers. Then

\[
\left| \sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} \right| \leq (m+1) \sin \frac{\pi}{2(m+1)} \left( \sum_{u=0}^{m} a_u^2 \right)^{1/2} \left( \sum_{u=0}^{m} b_u^2 \right)^{1/2}.
\]

**REFERENCES**