MATH. SCAND. 3 (1955), 281–282

NOTE ON A q-IDENTITY

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Fjeldstad [2] has proved the elegant formula

$$(1)\qquad \sum_{s=0}^{2m} (-1)^s \binom{2m}{s} \binom{2n}{n-m+s} \binom{2p}{p-m+s} = (-1)^m \frac{(m+n+p)!\,(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(m+p)!\,(n+p)!\,m!\,n!\,p!}.$$

An equivalent form of (1) is

$$(2)\qquad \sum_{s} (-1)^s \binom{m+n}{m+s} \binom{n+p}{n+s} \binom{p+m}{p+s} = \frac{(m+n+p)!}{m!\,n!\,p!}$$

where the summation is over all s yielding summands $\neq 0$.

It may be of interest to give the q-analog of (1) or (2). Put

$$(a)_r = (1-a)(1-qa)\ldots(1-q^{r-1}a), \qquad (a)_0 = 1,$$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]!\,[n-r]!}, \qquad [r]! = (q)_r .$$

Jackson has proved the identity [3, formula (2)]

$$(3)\qquad \sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} (a)_r (b)_r (a)_{2m-r} (b)_{2m-r} q^{\frac{1}{2}r(2m-r+1)} = (a)_m (b)_m (q^{m+1})_m (abq^m)_m .$$

In (3) take $a = q^{-m-n}$, $b = q^{-m-p}$. Then after a little manipulation we get

$$(4)\qquad \sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} \begin{bmatrix} 2n \\ n-m+r \end{bmatrix} \begin{bmatrix} 2p \\ p-m+r \end{bmatrix} q^{\frac{1}{2}\{3(r-m)^2+(r-m)\}} = (-1)^m \frac{[m+n+p]!\,[2m]!\,[2n]!\,[2p]!}{[m+n]!\,[m+p]!\,[n+p]!\,[m]!\,[n]!\,[p]!}.$$

Received November 15, 1955.

In the left member of (4) replace r by $m+s$; we get

$$\sum_{s=-m}^{m}(-1)^s \begin{bmatrix} 2m \\ m+s \end{bmatrix} \begin{bmatrix} 2n \\ n+s \end{bmatrix} \begin{bmatrix} 2p \\ p+s \end{bmatrix} q^{\frac{1}{2}(3s^2+s)} = \frac{[m+n+p]!\,[2m]!\,[2n]!\,[2p]!}{[m+n]!\,[m+p]!\,[n+p]!\,[m]!\,[n]!\,[p]!}. \tag{5}$$

An equivalent form of (5) is

$$\sum_{s=-m}^{m}(-1)^s \begin{bmatrix} m+n \\ m+s \end{bmatrix} \begin{bmatrix} n+p \\ n+s \end{bmatrix} \begin{bmatrix} p+m \\ p+s \end{bmatrix} q^{\frac{1}{2}(3s^2+s)} = \frac{[m+n+p]!}{[m]!\,[n]!\,[p]!}. \tag{6}$$

The formulas (4) and (6) may be compared with (1) and (2), respectively.
When $m=n=p$, (4) reduces to

$$\sum_{r=0}^{2m}(-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix}^3 q^{\frac{1}{2}\{3(r-m)^2+(r-m)\}} = (-1)^m \frac{[3m]!}{\{[m]!\}^3},$$

an identity noted by Bailey [1].

REFERENCES

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3. F. H. Jackson, *Certain q-identities*, Quart. J. Math., Oxford Ser., 12 (1941), 167–172.

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