ON NON-CONSTRUCTIVE THEOREMS OF ANALYSIS
AND THE DECISION PROBLEM

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In this note we exhibit a method for strengthening some known results on the impossibility of proving certain classical theorems in recursive analysis. For the concepts and nomenclature of recursive analysis the reader is referred to [3].

We denote by $\mathcal{R}$ some (unspecified) formalisation of recursive arithmetic, and by $\mathcal{R}^*$ an extension of $\mathcal{R}$ to rational numbers and functions, adequate for recursive analysis.

1. If $f(n, x)$ is any rational recursive function, recursively convergent in $n$, and differentiable in $x$, relative to $n$, with relative derivative $f^1(n, x)$ for $0 \leq x \leq 1$, and if

$$f(n, 0) = f(n, 1) = 0$$

then we say that $f(n, x)$ satisfies the conditions of the relative Rolle’s theorem and write $f \in RT$.

It was proved in [1] that if $f \in RT$ then there is a recursive $n$ and a recursive $c_k$ such that $n \geq n_k \rightarrow f^1(n, c_k) = 0(k)$ is provable in $\mathcal{R}^*$.

If there exists a recursive sequence $c_k$, a recursive $v_k$ and an integer $p$ such that $1/p < c_k < 1 - 1/p$ and

$$(i) \quad n \geq n_k \rightarrow f^1(n, c_k) = 0(k)$$

is provable in $\mathcal{R}^*$ (with free variable $n$) then we say that $f(n, x)$ satisfies the conditions of the uniform Rolle’s theorem and write $f \in URT$; if however condition $(i)$ is provable, not necessarily for a variable $n$, but for each positive integral value of $n$, then we write $f \in IRT$.

We gave in [2] an example of a function $f$ such that $f \in RT$ but $f \notin URT$. We shall now prove the stronger result that there exists an $f$ such that $f \in RT$ but $f \notin IRT$. We shall in fact show that a proof in $\mathcal{R}^*$ of

$$f \in RT \rightarrow f \in IRT$$

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provides a decision method for the class of equations \( q(n) = 0 \), where \( q \) is a recursive function which takes only the values 0 and 1, but, as is well known (see [4, pp. 417–418]), this class of equations is undecidable.

2. Given any recursive function \( q(n) \) which takes only the values 0 and 1 we define (as in [2])

\[
e_0 = 0, \quad e_{n+1} = e_n + \prod_{r=0}^{n} (1 - q(r)) \]
\[
d_0 = 1, \quad d_{n+1} = 1/e_{n+1} \]

and, for \( 0 \leq x \leq 1 \) and \( n \geq 3 \),

\[
f(n, x) = \frac{d_n^4 x (1 - x)}{d_n^2 + (1 - 2d_n) x}. \]

It is supposed that \( q(0) = 0 \).

The following properties of these functions are readily provable in \( \mathcal{R}^* \) (for details, see [2, pp. 228–230]).

(2.1) \( e_n \leq n \).

(2.2) \( e_n < n \rightarrow (Er)(r \leq n \text{ & } q(r) = 1) \).

(2.3) If \( N > n \geq 1 \) then \( 0 \leq d_n - d_N < 1/n \).

(2.4) For \( n \geq 3 \) and \( 0 \leq x \leq 1 \), we have

\[
0 \leq f(n, x) \leq d_n^4. \]

(2.5) If \( 3 \leq n < N \) and \( 0 \leq x \leq 1 \) then

\[
0 \leq f(n, x) - f(N, x) < 1/n^4, \]

from which it follows that \( f(n, x) \) converges uniformly in \( x \) for \( 0 \leq x \leq 1 \).

(2.6) For \( 0 \leq x \leq 1 \), \( f(n, x) \) is differentiable in \( x \) uniformly in \( x \) and \( n \), so that \( f(n, x) \) is differentiable in \( x \) relative to \( n \), and the relative derivative \( f^1(n, x) \) converges uniformly in \( x \), and \( f \in RT \).

(2.7) If there is a recursive \( V(k) \) such that

\( n \geq V(k) \rightarrow d_n = 0(k) \)

is provable in \( \mathcal{R}^* \) for all integers \( n \), then \( q(n) = 0 \) is provable in \( \mathcal{R} \) for all integers \( n \).

3. If \( f \in IRT \) then (by definition) there exists a recursive \( V(k) \), a recursive \( c_k \) and an integer \( p \) such that \( c_k \geq 1/p \) and

\[
n \geq V(k) \rightarrow f^1(n, c_k) = 0(k) \]

is provable in \( \mathcal{R}^* \) for all integers \( n \).
Since
\[ f^1(n, c_k) = \frac{d_n^4(d_n - c_k) \{d_n + (1 - 2d_n)c_k\}}{\{d_n^2 + (1 - 2d_n)c_k\}^2} \]
and
\[ d_n + (1 - 2d_n)c_k > d_n^2 + (1 - 2d_n)c_k, \]
\[ d_n^2 + (1 - 2d_n)c_k \leq (1 - d_n)^2 < 1, \]
it follows that
\[ n \geq V(k) \rightarrow d_n^4(d_n - c_k) = 0(k) \]
is provable in $R^*$ for all $n$.
By (2.1), either $d_{p+1} = 1/(p + 1)$ or $d_{p+1} > 1/(p + 1)$.
If $d_{p+1} = 1/(p + 1)$ then $d_n \leq 1/(p + 1)$ for $n \geq (p + 1)$ so that
\[ |d_n - c_k| > 1/[p(p + 1)] , \]
and therefore
\[ n \geq V(4k + p) \rightarrow d_n = 0(k) \]
is provable in $R^*$ for all $n$, whence, by (2.7), $\varrho(n) = 0$ is provable in $R$ for
all integers $n$.
If however $d_{p+1} > 1/(p + 1)$ then by (2.2) there is an $r$ between $0$ and
$p + 1$ for which $\varrho(r) = 1$ is provable in $R$, and so the hypothesis $f \in IRT$
implies the existence of a decision procedure for the undecidable class
of equations $\varrho(n) = 0$.

**BIBLIOGRAPHY**


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