ESTIMATES OF THE FRIEDRICHS-LEWY TYPE
FOR A HYPERBOLIC EQUATION
WITH THREE CHARACTERISTICS

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The aim of this paper is to prove the uniqueness of a solution of a boundary value problem of mixed type for a linear hyperbolic differential equation with three characteristics and to estimate its solution by means of the boundary values. Problems involving more than two characteristics occur in gas dynamics, and uniqueness theorems of the kind considered below are stated without proofs in [1] (see e.g. p. 86). The method used in this paper is due to Friedrichs and Lewy [2], and has recently been used in the theory of linear hyperbolic equations of order greater than two by Leray [5] and Gårding [3]. For the sake of simplicity we have only considered the case when the differential operator has real coefficients and its principal part constant coefficients. Certain restrictions are also made concerning the boundary of the region considered. These specializations, however, do not seem to be imposed by the problem, but are made in order to avoid difficulties.

I wish to express my gratitude to my teacher, professor Lars Gårding, who suggested the subject of this paper.

1. Let \( V \) be a closed region in the plane whose boundary \( S \) is piecewise smooth and let \( L \) be a real hyperbolic differential operator of order three with constant principal part and continuous coefficients. In a suitable coordinate system, a suitable real multiple of such an operator has the form

\[
(1) \quad L = (D_x^2 - \alpha_1 D_1^2)(D_x^2 - \alpha_2 D_1^2)(D_x^2 - \alpha_3 D_1^2) + \sum_{i, k=1}^{2} a_{ik} D_i D_k + \sum_{i=1}^{2} b_i D_i + c
\]

where \( D_i = \partial / \partial x_i \); the constants \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are real and \( \alpha_1 < \alpha_2 < \alpha_3 \) and the coefficients \( a_{ik}, b_i \) and \( c \) are real-valued continuous functions in \( V \). The characteristic form associated with (1),

\[
(2) \quad A = A(\xi) = (\xi_2 - \alpha_1 \xi_1)(\xi_2 - \alpha_2 \xi_1)(\xi_2 - \alpha_3 \xi_1),
\]

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divides the $\xi$-space (the dual space) into four parts $\Sigma_i$ ($i = 0, 1, 2, 3$), where $\Sigma_i$ is the set of points $\xi = (\xi_1, \xi_2)$ making exactly $i$ factors of (2) $< 0$ and the others $\geq 0$. Since every factor $\xi_2 - \alpha_k \xi_1$ is negative on the negative $\xi_2$-axis, the division is seen to be the one represented in fig. 1. For practical reasons we also distinguish between the parts of $\Sigma_1$ and $\Sigma_2$ which correspond to positive and negative values of $\xi_1$ and denote these parts by $\Sigma_1^-, \Sigma_1^+, \Sigma_2^-$ and $\Sigma_2^+$, respectively.

This division of the dual space now gives rise to a division of $S$ into parts $S_i$ so that a point of $S$ belongs to $S_i$ if and only if the exterior normal $\nu$ of $S$ in this point belongs to $\Sigma_i$ ($\nu = (\nu_1, \nu_2); \ |\nu| = (\nu_1^2 + \nu_2^2)^{\frac{1}{2}} = 1$). The notations $S_1^-, S_1^+, S_2^-$ and $S_2^+$ correspond in an obvious way to $\Sigma_1^-, \Sigma_1^+, \Sigma_2^-$ and $\Sigma_2^+$ (see fig. 2).

Certain gas-dynamical considerations (see e.g. [1]) make it probable that, roughly speaking, if $\varphi$ is a given function in $V$ and $\varphi_{k_i}$ ($0 \leq k < i$) are given functions on $S_i$ ($i = 1, 2, 3$), the differential equation

$$Lu = \varphi$$

will have a unique solution $u$ such that $d^k u/d\nu^k = \varphi_{k_i}$ on $S_i$ when $k < i$ ($i = 1, 2, 3$). We shall prove the uniqueness part of this statement under the following restrictions on the boundary $S$ of $V$,

(a) $$\inf_{\Sigma_1 \cup \Sigma_2} |\lambda_k| > 0 \quad (k = 1, 2, 3)$$

where $\lambda_k = \nu_2 - \alpha_k \nu_1$ and

(b) $S_1^+ \cup S_2^+$ has a positive distance to $S_1^- \cup S_2^-$. 

The first of these conditions means that on $S_1$ and $S_2$, the normal must
avoid the shaded areas in fig. 1 in the neighbourhoods of the lines 
\( \xi_2 - \alpha_k \xi_1 = 0 \) \( (k = 1, 2, 3) \). Observe that this implies that a passage from 
one part \( S_i \) of \( S \) to another is accompanied by a jump in the normal 
derivative. The condition (b) means that any part of \( S_1^- \cup S_2^- \) is separated 
from any part of \( S_1^+ \cup S_2^+ \) by a part of \( S_0 \) or \( S_3 \).

A solution \( u \) of (3) is to be understood here as a real function whose 
derivatives of order \( \leq 3 \) are continuous in \( V \).

**Theorem 1.** Let \( V \) be a region whose boundary \( S \) is piecewise smooth 
and satisfies (a) and (b). Then a solution of the equation

\[
Lu = 0
\]

with the boundary conditions \( \frac{d^k u}{dx^k} = 0 \) for \( k < i \) on \( S_i \) \( (i = 0, 1, 2, 3) \) 
vansishes identically in \( V \).

In the proof we shall introduce a new differential operator

\[
M = (D_2 - \beta_1 D_1)(D_2 - \beta_2 D_1)
\]

where \( \beta_1 \) and \( \beta_2 \) are continuously differentiable functions satisfying 
\( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \)
at each point of \( V \). They will be subject to additional conditions later. 
First we shall prescribe their values on the different parts of the boundary 
\( S \) in a suitable way, and then choose \( \beta_1 \) and \( \beta_2 \) as continuously differentiable 
continuations of these values to the whole of \( V \).

The notations

\[
l_k = D_2 - \alpha_k D_1 \quad (k = 1, 2, 3)
\]

and

\[
L_k = \prod_{i+k} (D_2 - \alpha_i D_1) \quad (i, k = 1, 2, 3)
\]

will be of constant use.

We first prove the following

**Lemma.** Suppose that \( L \) only contains the principal part 
\( (D_2 - \alpha_1 D_1)(D_2 - \alpha_2 D_1)(D_2 - \alpha_3 D_1) \)

and that the \( \beta_i \) of (5) are constants so that \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \). Then we have the identity

\[
2 Lu M u = \sum_{k=1}^{3} l_k A_k (L_k u)^2
\]

where \( A_k \) are positive constants.
Proof. The relation
\[ Mu = \sum_{k=1}^{3} A_k L_k u \]
is evidently valid if the constants \( A_k \) satisfy the system
\[
A_1 + A_2 + A_3 = 1, \\
A_1(x_2 + \alpha_3) + A_2(\alpha_3 + \alpha_1) + A_3(\alpha_1 + \alpha_2) = \beta_1 + \beta_2, \\
A_1 \alpha_2 \alpha_3 + A_2 \alpha_3 \alpha_1 + A_3 \alpha_1 \alpha_2 = \beta_1 \beta_2.
\]
which has the unique solution
\[
\begin{aligned}
A_1 &= (\beta_1 - \alpha_1)(\beta_2 - \alpha_1)/(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1), \\
A_2 &= (\beta_1 - \alpha_2)(\beta_2 - \alpha_2)/(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2), \\
A_3 &= (\beta_1 - \alpha_3)(\beta_2 - \alpha_3)/(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3).
\end{aligned}
\]
Here \( A_k \) are seen to be positive if \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \). Now multiplying (8) by \( 2Lu \), we get
\[
2Lu \, Mu = \sum_{k=1}^{3} 2A_k \, L_k u \, Lu = \sum_{k=1}^{3} 2A_k \, L_k u \, l_k \, L_k u = \sum_{k=1}^{3} l_k A_k (L_k u)^2
\]
which proves the lemma.

To prove Theorem 1 we observe that even if \( Lu \) contains derivatives of lower order and \( \beta_1 \) and \( \beta_2 \) are not constants but continuously differentiable functions of \( x \) on \( V \), the difference between the right and left sides of (7) does not contain derivatives of order three, i.e. we have
\[
2Lu \, Mu = \sum_{k=1}^{3} l_k A_k (L_k u)^2 + R(u, u)
\]
where \( R(u, u) \) is a quadratic form in \( u_{ik} = D_i D_k u, \, u_i = D_i u \) and \( u \) with continuous coefficients.

Following Hörmander [4] we multiply (10) by a weight function \( e^{-\gamma x_1} \), where the constant \( \gamma > 0 \) will be chosen later. After easy computations we get
\[
2e^{-\gamma x_1} Lu \, Mu
= \sum_{k=1}^{3} l_k \left[ e^{-\gamma x_1} A_k (L_k u)^2 \right] + e^{-\gamma x_1} \left[ \gamma \sum_{k=1}^{3} A_k (L_k u)^2 + R(u, u) \right].
\]
One immediately verifies the identity
\[
0 = l_2 \left[ e^{-\gamma x_1} (u_1^2 + u_2^2 + u^2) \right] - e^{-\gamma x_1} l_2 (u_1^2 + u_2^2 + u^2) + \\
\quad + \gamma e^{-\gamma x_1} (u_1^2 + u_2^2 + u^2).
\]
Multiplying (12) by a non-negative constant $\delta$ and adding (11) we get

\begin{equation}
2e^{-\gamma x_2} Lu \cdot Mu \\
= \sum_{k=1}^{3} l_k \left[ e^{-\gamma x_2} A_k (L_k u)^2 \right] + l_2 \left[ e^{-\gamma x_2} \delta (u_1^2 + u_2^2 + u_3^2) \right] + e^{-\gamma x_2} \left[ \gamma V(u, u) + Q(u, u) \right]
\end{equation}

where

\begin{equation}
V(u, u) = \sum_{k=1}^{3} A_k (L_k u)^2 + \delta (u_1^2 + u_2^2 + u_3^2)
\end{equation}

and $Q(u, u)$ is a quadratic form in $u_{ik}$, $u_i$ and $u$ with continuous coefficients.

Integrating (13) over $V$ and using Greens formula we get

\begin{equation}
\int_{V} 2e^{-\gamma x_2} Lu \cdot Mu \, dV = \int_{S} e^{-\gamma x_2} S(u, u) \, dS + \int_{V} e^{-\gamma x_2} [\gamma V(u, u) + Q(u, u)] \, dV
\end{equation}

where

\begin{equation}
S(u, u) = \sum_{k=1}^{3} \lambda_k A_k (L_k u)^2 + \lambda_2 \delta (u_1^2 + u_2^2 + u_3^2).
\end{equation}

For a solution of (4) the left side of (15) vanishes and we shall prove that $S(u, u)$ and $V(u, u)$ can be made positive definite on $S$ and in $V$ respectively by a suitable choice of the functions $\beta_1$ and $\beta_2$ and the constant $\delta$. In this way, if $\gamma$ is sufficiently large, we have written the right side of (15) as a sum of two non-negative terms, which must therefore both vanish. The vanishing of the second term combined with the positive definiteness of the integrand implies that $u$ vanishes in $V$, which is what we wanted to prove.

We start with $S(u, u)$ and consider first the various parts of $S$ separately.

By virtue of the boundary conditions, $S(u, u)$ vanishes identically on $S_3$ and thus we can choose $\beta_1$, $\beta_2$ and $\delta$ arbitrarily on $S_3$. As $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$ the $A_k$ are positive and according to the definition of $S_0$ the $\lambda_k$ are positive on $S_0$. Therefore, as $\delta$ is chosen positive, we have that $S(u, u)$ is non-negative on $S_0$. On $S_1$ and $S_2$, $S(u, u)$ will be proved to be positive definite in the derivatives which do not vanish according to the boundary conditions. Thus we express $S(u, u)$ in terms of $u$ and its derivatives in $S$ and orthogonal to $S$. After the transformation

\[
D_s = v_1 D_2 - v_2 D_1 \\
D_r = v_1 D_1 + v_2 D_2
\]

we obtain from (16) after easy computations
\[ S(u, u) = \lambda_1 \lambda_2 \lambda_3 \mu_1 \mu_2 \left\{ \sum_{k=1}^{3} A_k' (L_k' u)^2 + \lambda_2 \delta (u_e^2 + u_r^2 + u^2) + W(u, u) \right\} \]

where \( W(u, u) \) is a quadratic form which is linear in \( u_{es}, u_{ev} \) and \( u_{rv}, \mu_k = \nu_k - \beta_k v_1, \) and \( L_k' \) and \( A_k' \) are the expressions corresponding to (6) and (9) with \( \alpha_i^j \) and \( \beta_i^j \) replaced by

\[ \alpha_i' = (v_1 + \alpha_i^j v_2)/(\alpha_i^j v_1 - v_2) \]

and

\[ \beta_i' = (v_1 + \beta_i^j v_2)/(\beta_i^j v_1 - v_2), \]

respectively.

Here \( \alpha_i^j v_1 - v_2 = -\lambda_i \) is never zero on \( S_1 \) and \( S_2 \) because of the condition (a), and \( \beta_1 \) and \( \beta_2 \) are to be chosen in a way which makes \( \beta_i^j v_1 - v_2 = -\mu_i \) different from zero on \( S_1 \) and \( S_2 \).

On \( S_2 \) we have \( u = u_e = u_r = u_{es} = u_{ev} = 0 \) and we get

\[ S(u, u) = \lambda_1 \lambda_2 \lambda_3 \mu_1 \mu_2 u_{rv}^2. \]

To make the coefficient of \( u_{rv}^2 \) positive on \( S_2^+ \) we have only to choose \( \beta_1 \) sufficiently close to \( \alpha_2 \) and \( \beta_2 \) arbitrarily in the interval \( (\alpha_2, \alpha_3) \). To see this we notice that for an arbitrary point of \( S_2^+ \) we have \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0 \) and \( \mu_2 < 0 \). Thus it suffices to choose \( \beta_1 \) in a way which makes \( \mu_1 < 0 \) for all points on \( S_2^+ \). Now choosing \( \beta_1 \) sufficiently close to \( \alpha_2 \) we obtain that the part of the line \( \xi_2 - \beta_1 \xi_1 = 0 \) which corresponds to positive values of \( \xi_1 \) belongs to the shaded area of \( \Sigma_2^+ \) close to the line \( \xi_2 - \alpha_2 \xi_1 = 0 \) in fig. 1. That means that for all allowed \( \nu \) on \( S_2^+ \) we have \( \mu_1 < 0 \). Similarly, to make the coefficient of \( u_{rv}^2 \) positive on \( S_2^- \) we have only to choose \( \beta_2 \) sufficiently close to \( \alpha_2 \) and \( \beta_1 \) arbitrarily in the interval \( (\alpha_1, \alpha_2) \). The proof is analogous to that for \( S_2^+ \).

On \( S_1 \) we have \( u = u_e = u_{es} = 0 \). We shall prove that after having fixed \( \beta_1 \) arbitrarily in the interval \( (\alpha_1, \alpha_3) \) we have only to choose \( \beta_1 \) sufficiently close to \( \alpha_3 \) in order to make \( S(u, u) \) positive definite on \( S_1^+ \) and similarly that after having fixed \( \beta_2 \) arbitrarily in the interval \( (\alpha_2, \alpha_3) \) we have only to choose \( \beta_2 \) sufficiently close to \( \alpha_1 \) in order to make \( S(u, u) \) positive definite on \( S_1^- \). It is sufficient to prove this for \( S_1^+ \), the proof for \( S_1^- \) being analogous.

We first restrict ourselves to such \( \beta_2 \) which satisfy

\[ \lambda_1 \lambda_2 \lambda_3 \mu_1 \mu_2 > 0 \]

Similarly to the discussion on \( S_2^+ \) above, we realize that this is obtained if we choose \( \beta_2 \) sufficiently close to \( \alpha_3 \). In a fixed point of \( S_1^+ \) we have

\[ \beta_2' < \alpha_3' < \alpha_1' < \beta_1' < \alpha_2'. \]
if \( \beta_2 \) is sufficiently close to \( \alpha_3 \), and we start by proving that in this fixed point \( S(u, u) \) can be made positive definite. Now we notice that choosing \( \beta_2 \) close to \( \alpha_3 \) is equivalent to choosing \( \beta_2' \) close to \( \alpha_3' \). The main part of \( S(u, u) \) is

\[
\sum_{k=1}^{3} A_k' (L_k' u)^2 = u_{vv}^2 - 2(\beta_1 + \beta_2) u_{sv} u_{rr} + [A_1' (\alpha_2' + \alpha_3') + A_2' (\alpha_3' + \alpha_1') + A_3' (\alpha_1' + \alpha_2')] u_{sv}^2 = [u_{vv} - (\beta_1' + \beta_2') u_{sv}]^2 + D(\beta_1', \beta_2') u_{sv}^2
\]

where

\[
D(\beta_1', \beta_2') = A_1' (\alpha_2' + \alpha_3') + A_2' (\alpha_3' + \alpha_1') + A_3' (\alpha_1' + \alpha_2') - (\beta_1' + \beta_2')^2.
\]

For \( \alpha_1' < \beta_1' < \alpha_2' \) we get

\[
D(\beta_1', \alpha_3') = (\beta_1' - \alpha_1')(\alpha_2' - \beta_1') > 0.
\]

Since \( D(\beta_1', \beta_2') \) is a continuous function of \( \beta_2' \) in the neighbourhood of \( \alpha_3' \), it follows that \( D(\beta_1', \beta_2') \) is positive in a sufficiently small neighbourhood of \( \alpha_3' \). For a fixed \( \beta_1 \) we have thus found that for each point of \( S_1^+ \) there exists a neighbourhood of \( \alpha_3 \) in which

\[
(17) \quad \sum_{k=1}^{3} A_k' (L_k' u)^2
\]

is positive definite for all \( \beta_2 \). By continuity it follows that we can choose \( \beta_2 \) as a constant so that (17) is positive definite in \( u_{sv} \) and \( u_{sv} \) on the whole of \( S_1^+ \). Choosing \( \delta \) sufficiently large we obtain that

\[
\sum_{k=1}^{3} A_k' (L_k' u)^2 + \lambda_2 \delta u_{sv}^2 + W(u, u)
\]

is positive definite in \( u_{sv}, u_{sv} \) and \( u_r \) on \( S_1^+ \), and thus \( S(u, u) \) is positive definite on \( S_1^+ \).

We have now actually proved that we can choose \( \beta_1 \) and \( \beta_2 \) as constants on \( S_1^+ \cup S_2^+ \) and \( S_1^- \cup S_2^- \), respectively. Take for instance \( S_1^+ \cup S_2^+ \), the reasoning for \( S_1^- \cup S_2^- \) being analogous. Then we first fix \( \beta_1 \) sufficiently close to \( \alpha_2 \) to get \( S(u, u) \) positive definite on \( S_2^+ \) and after that we choose \( \beta_2 \) so that \( S(u, u) \) becomes positive definite on \( S_1^+ \).

Since \( S_1^+ \cup S_2^+ \) has a positive distance to \( S_1^- \cup S_2^- \) it is evident that there exists continuously differentiable functions \( \beta_1 \) and \( \beta_2 \) in \( V \) taking the chosen constant values on \( S_1^+ \cup S_2^+ \) and \( S_1^- \cup S_2^- \) and which in \( V \) attain only values between the values attained on \( S_1^+ \cup S_2^+ \) and \( S_1^- \cup S_2^- \).

This choice of \( \beta_1 \) and \( \beta_2 \) guarantees the existence of an \( \varepsilon > 0 \) so that

\[
\alpha_1 + \varepsilon \leq \beta_1 \leq \alpha_2 - \varepsilon
\]

and

\[
\alpha_2 + \varepsilon \leq \beta_2 \leq \alpha_3 - \varepsilon.
\]
This now implies that $\inf_V A_k > 0$ ($k = 1, 2, 3$), that is, the quadratic form $V(u, u)$ defined in (14) is positive definite in $V$. This completes the proof of Theorem 1.

2. Finally we prove that it is possible to obtain an estimate for the solution of the equation (3) in terms of $\varphi = Lu$ and the boundary values.

**Theorem 2.** For a solution of (3) in a region $V$ whose boundary $S$ is submitted to the same restrictions as in Theorem 1 we have the following estimate

\[
(18) \int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV \leq C \left\{ \int_V \varphi^2 dV + \int_{S_1} (u_{ss}^2 + u_s^2 + u^2) dS + \int_{S_2} (u_{sr}^2 + u_r^2 + u_s^2 + u^2) dS + \int_{S_3} (u_{ss}^2 + u_{sr}^2 + u_{rr}^2 + u_r^2 + u_s^2 + u^2) dS \right\}
\]

where $C$ is a constant independent of the function $u$.

**Proof.** Having chosen the functions $\beta_1$ and $\beta_2$ and the constants $\gamma$ and $\delta$ as in the proof of Theorem 1 we get from (15) (observe that $C$ does not necessarily denote the same constant during the course of the proof)

\[
\int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV \leq C \left\{ \int_V Lu Mu dV - \int_S (u, u) dS \right\}
\]

\[
\leq C \left\{ \left[ \int_V (Lu)^2 dV \int_V (Mu)^2 dV \right]^{\frac{1}{2}} - \int_S (u, u) dS \right\}
\]

\[
\leq C \left\{ \left[ \int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV \int_V \varphi^2 dV \right]^{\frac{1}{2}} - \int_S (u, u) dS \right\}
\]

\[
= \left[ \int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV \int_V C^2 \varphi^2 dV \right]^{\frac{1}{2}} - C \int_S (u, u) dS
\]

\[
\leq \frac{1}{2} \int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV + \frac{1}{2} C^2 \int_V \varphi^2 dV - C \int_S (u, u) dS,
\]

that is,

\[
\int_V \left( \sum_{i, k=1}^2 u_{ik}^2 + \sum_{i=1}^2 u_i^2 + u^2 \right) dV \leq C \left\{ \int_V \varphi^2 dV - \int_S (u, u) dS \right\}.
\]
The inequality (18) is now obtained by estimating

\[- \int_S S(u, u) \, dS\]

on the various parts of $S$, and this is easily done, using that $S(u, u)$ is positive definite when the boundary values vanish.

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