THE MAXIMUM VALUE OF A FOURIER-STIELTJES TRANSFORM

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1. Introduction. Let $G$ be a locally compact Abelian group with character group $G^*$. Let $(y, x)$ denote the function on $G \times G^*$ equal to the value of $y \in G^*$ at $x \in G$. Let $\varphi$ be a bounded Radon measure on $G$, with Fourier-Stieltjes transform

$$\Phi(y) = \int_G (y, x) \, d\varphi(x).$$

Let $|\varphi|$ be the total variation of the measure $\varphi$ (see [2, p. 459]). That is,

$$|\varphi|(A) = \sup \sum_{r=1}^n |\varphi(A_r)|,$$

the supremum being taken over all pairwise disjoint families $\{A_r\}_{r=1}^n$ of Borel sets whose union is the Borel set $A$. For other notation and terminology, see [3].

We are concerned in this note with the sets

$$A(\varphi) = E\left[ y; \quad y \in G^*, \quad |\Phi(y)| = \int_G d|\varphi|(x) \right]$$

and

$$M(\varphi) = E\left[ y; \quad y \in G^*, \quad \Phi(y) = \int_G d|\varphi|(x) \right].$$

We shall establish the following results, which characterize the possible sets $A(\varphi)$ and $M(\varphi)$ completely.

1.1 Theorem. The following conditions on a subset $E$ of $G^*$ are equivalent:

1.1.1 $E$ has the form $A(\varphi)$ for some bounded Radon measure $\varphi$;

1.1.2 $E$ has the form $M(\varphi)$ for some bounded Radon measure $\varphi$;

1.1.3 $E$ contains a non-void $G_0$ and is a closed subgroup of $G^*$ or is a translate of such a subgroup, or $E = 0$. 

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1.2 Theorem. The measure \( \varphi \) referred to in Theorem 1.1 can be chosen to be absolutely continuous with respect to Haar measure on \( G \) if and only if \( E \) is compact.

These theorems appear to be known for the case \( G = G^* = \) the real line \( R \) under addition. The only compact subgroup of \( R \) being \( \{0\} \), Theorem 1.2 implies that integrals

\[
\int_{-\infty}^{\infty} e^{ixv} \, d\varphi(x)
\]

for non-negative absolutely continuous measures \( \varphi \) are equal to \( \varphi(R) \) for \( y = 0 \) and are less than \( \varphi(R) \) in absolute value for \( y \neq 0 \). The only proper closed subgroups \( H \) of \( R \) are of the form \( \{ n \alpha \}_{n=-\infty}^{\infty} \) \((\alpha \in \mathbb{R})\). The measure \( \varphi = \frac{1}{2} \varepsilon_{n/\alpha} + \frac{1}{2} \varepsilon_{-n/\alpha} \) has the property that \( A(\varphi) = H \). The measure \( \varphi = \frac{1}{4}[\varepsilon_{0} + \varepsilon_{2n/\alpha}] \) has the property that \( M(\varphi) = H \). These observations show that 1.1.3 is sufficient for 1.1.1 and 1.1.2 in the case \( G = R \).

2. Measure-theoretic observations. We first prove some simple properties of \( M(\varphi) \) and \( A(\varphi) \).

2.1 Theorem. Let \( \varphi \) be a bounded Radon measure on \( G \). Then there exists a bounded complex-valued Borel-measurable function \( h \) on \( G \) such that

\[
d\varphi(x) = h(x) \, d|\varphi|(x) .
\]

Proof. Since \( \varphi = \varphi_1 + i\varphi_2 \), where \( \varphi_j \) is a real-valued measure and since \( \varphi_j \) is absolutely continuous with respect to \( |\varphi| \), we can apply the Radon-Nikodym theorem ([1, p. 129 et. seq.]) to write

\[
d\varphi_j(x) = h_j(x) \, d|\varphi|(x) \quad (j = 1, 2) .
\]

We then take \( h = h_1 + ih_2 \).

2.2 Theorem. Let \( \varphi \) be a bounded Radon measure on \( G \), and let \( y_0 \) denote an element of \( G^* \). Then

\[
y_0 \in M(\varphi) \quad \text{if and only if} \quad d\varphi(x) = (y_0^{-1}, x) \, d\pi(x) ,
\]

where \( \pi \) is a non-negative bounded Radon measure on \( G \).

Proof. It follows from 2.1 that

\[
\int_{G} (y_0, x) \, d\varphi(x) = \int_{G} (y_0, x) \, h(x) \, d|\varphi|(x) .
\]
It is easy to see that the relation
\[ \int_G (y_0, x) h(x) \, d|\varphi|(x) = \int_G 1 \cdot d|\varphi|(x), \]
holds if and only if \( (y_0, x) h(x) = \text{ess sup} \, |h(x)| \) almost everywhere with respect to \(|\varphi|\), the ess sup being taken with respect to \(|\varphi|\). (See for example [4, Theorem 3.1.].) It follows that \( h(x) = \beta (y_0^{-1}, x) \) with a positive constant \( \beta \) almost everywhere with respect to \(|\varphi|\). Hence we take \( \pi = \beta |\varphi| \).

2.3 Theorem. Let \( \varphi \) be a bounded Radon measure on \( G \), and let \( y_0 \) be an element of \( G^* \). Then
\[ y_0 \in A(\varphi) \quad \text{if and only if} \quad d\varphi(x) = \alpha (y_0^{-1}, x) \, d\pi(x), \]
where \( \pi \) is a non-negative bounded Radon measure on \( G \) and \(|x| = 1\).

Proof. We have
\[ \int_G (y_0, x) \, d\varphi(x) = \int_G d|\varphi|(x) \]
if and only if there is a number \( \delta \) of absolute value 1 such that
\[ \int_G (y_0, x) \, d[\delta \varphi](x) = \int_G d|\delta \varphi|(x); \]
this brings us back to Theorem 2.2, and we may take \( \alpha = \delta^{-1} \).

2.4 Theorem. Suppose that \( M(\varphi) \neq 0 \). Then the set \( M(\varphi) \) is a closed \( G_\delta \) and is a subgroup or a translate of a subgroup of \( G^* \).

Proof. Since \( M(\varphi) \) is the set where the continuous function \( \Phi \) assumes a fixed value, it is clearly a closed \( G_\delta \). It remains only to show that it is a subgroup or a translate of a subgroup. Applying Theorem 2.2, we can multiply \( d\varphi \) by a character and suppose that the measure \( \varphi \) is non-negative and that accordingly the identity of \( G^* \) lies in \( M(\varphi) \). This of course amounts simply to translating \( M(\varphi) \). Under these hypotheses, \( y \in M(\varphi) \) if and only if \( (y, x) = 1 \) almost everywhere with respect to \( \varphi \). The set of such \( y \) clearly forms a subgroup of \( G^* \).

2.5 Theorem. Suppose that \( A(\varphi) \neq 0 \). Then \( A(\varphi) \) is a closed \( G_\delta \) which is either a subgroup of \( G^* \) or a translate of a subgroup of \( G^* \).

Proof. Similar to the proof of Theorem 2.4.

Theorems 2.4 and 2.5 show that 1.1.3 is necessary for 1.1.1 and 1.1.2.
3. Group-theoretic observations. Let $S$ be a subset of $G$. The set $N(S)$, the annihilator of $S$, is the set of all $y \in G^*$ such that $(y, x) = 1$ for all $x \in S$. It is obvious that $N(S)$ is a closed subgroup of $G^*$ and it is well known that $N(N(S)) = S$ if $S$ is a closed subgroup of $G$.

3.1 Theorem. Let $S$ be a subset of $G$. If $S$ contains a non-void $G_\delta$, then $N(S)$ is $\sigma$-compact.

Proof. Let $\{Q_n\}_{n=1}^\infty$ be a sequence of open subsets of $G$ such that

$$0 = \bigcap_{n=1}^\infty Q_n = T \subset S.$$ 

Let $x$ be any element of $T$. Then there exists a sequence of open sets $\{U_n\}_{n=1}^\infty$ such that $x \in U_n$, $U_n \subset Q_n$, $U_n^-$ is compact ($n = 1, 2, 3, \ldots$) and $U_n^- \subset U_{n-1}$ ($n = 2, 3, 4, \ldots$). Now let $\delta$ be a positive real number less than $\frac{1}{10}$, and let

$$V_n = E[y; ~ y \in G^*, |(y, x) - 1| < \delta \text{ for all } x \in U_n^-]$$

($n = 1, 2, 3, \ldots$). It is known that $V_n^-$ is compact in $G^*$. Hence

$$W = \bigcup_{n=1}^\infty V_n^-$$

is $\sigma$-compact. We now show that $N(S) \subset W$. In fact, if

$$y \in N(S) \cap W',$$

then for every positive integer $n$, there exists $x_n \in U_n^-$ such that

$$|(y, x_n) - 1| \geq \delta.$$ 

Since $x_n \in U_m^-$ for $m \leq n$, there is a point

$$x_0 \in \bigcap_{n=1}^\infty U_n^- \subset T \subset S$$

such that every neighborhood of $x_0$ contains an infinite number of the points $x_n$. It follows that $|(y, x_0) - 1| \geq \delta$, and this is inconsistent with the relation $y \in N(S)$. Thus $N(S)$ is contained in a $\sigma$-compact set. Since $N(S)$ is closed, it follows that $N(S)$ is $\sigma$-compact.

We note also that the annihilator of a $\sigma$-compact subgroup of $G$ is a $G_\delta$.

3.2 Corollary. A closed subgroup of $G$ contains a non-void $G_\delta$ if and only if it is a non-void $G_\delta$.

3.3 Theorem. Let $H$ be a closed $\sigma$-compact subgroup of $G$. Then there exists a non-negative bounded Radon measure $\varphi$ on $G$ such that:
3.3.1 \( \varphi(A) > 0 \) for every non-void relatively open subset \( A \) of \( H \);
3.3.2 \( \varphi(H) = 1 \);
3.3.3 \( \varphi(H') = 0 \).

**Proof.** Let \( \lambda \) be a Haar measure on the group \( H \) (\( H \) is certainly a locally compact Abelian group). Since \( H \) is \( \sigma \)-compact, the measure \( \lambda \) is \( \sigma \)-finite. This implies that

\[
H = \bigcup_{n=1}^{\infty} P_n,
\]

where the sets \( P_n \) are pairwise disjoint and \( 0 < \lambda(P_n) < \infty \) \((n = 1, 2, 3, \ldots)\). Let the function \( f \) on \( H \) be defined by the relations

\[
f(x) = 2^{-n}[\lambda(P_n)]^{-1} \quad \text{for} \quad x \in P_n \quad (n = 1, 2, 3, \ldots).
\]

It is clear that \( f \in \mathcal{L}_1(H) \) and that

\[
\int_A f(x) \, d\lambda(x) > 0 \quad \text{if} \quad \lambda(A) > 0.
\]

For an arbitrary Borel set \( Q \subset G \), let

\[
\varphi(Q) = \int_{Q \cap H} f(x) \, d\lambda(x).
\]

It is obvious that this set-function satisfies all requirements of the present theorem.

3.4 **Theorem.** If the subgroup \( H \) of Theorem 3.3 is also open, then the measure \( \varphi \) of Theorem 3.3 can be taken as absolutely continuous with respect to Haar measure on \( G \).

**Proof.** This follows immediately from the fact that Haar measure on an open subgroup \( H \) of \( G \) is simply Haar measure on \( G \) relativized to \( H \).

3.5 **Remark.** Theorem 3.3 is not true for general locally compact \( \sigma \)-compact Hausdorff spaces. Let \( D \) denote a countably infinite discrete space and let \( \beta D \) denote the Stone-Čech compactification of \( D \). Then, as Nakamura and Kakutani have shown [5], the compact Hausdorff space \( \beta D \cap D' \) contains a continuum of pairwise disjoint non-void open sets. It is clear that no Borel measure on \( \beta D \cap D' \) can assign positive measure to every non-void open set.

4. **Completion of the proof of Theorem 1.1.** We shall now show that given a set \( E \subset G^* \) which contains a non-void \( G_\delta \) and is a closed subgroup
or a translate of such a subgroup, there exists a bounded Radon measure \( \varphi \) on \( G \) such that \( A(\varphi) = M(\varphi) = E \). Upon translating \( E \) if necessary, which is equivalent to multiplying \( d\varphi(x) \) by a character, we may suppose that \( E \) is a subgroup of \( G^\ast \). Now consider \( N(E) \subset G \). By Theorem 3.1, \( N(E) \) is a \( \sigma \)-compact subgroup of \( G \). Consider the measure \( \varphi \) described in Theorem 3.3, for \( H = N(E) \). Since \( N(N(E)) = E \), we have \( \Phi(y) = 1 \) for all \( y \in E \). Conversely, if \( |\Phi(y)| = 1 \) for an element \( y \) of \( G^\ast \), there exists a complex number \( \beta \) of absolute value 1 such that

\[
\int_G \beta(y, x) d\varphi(x) = 1,
\]

and \( \beta(y, x) = 1 \) almost everywhere with respect to \( \varphi \). Accordingly, \((y, x) = \beta^{-1}\) for all \( x \in N(E) \), and as \((y, e) = 1 \) (\( e \) the identity of \( G \)), we find \( \beta = 1 \) and \( y \in N(N(E)) = E \). This proves that \( |\Phi(y)| < 1 \) for \( y \notin E \), and establishes Theorem 1.1.

To prove Theorem 1.2, we note that if \( E \) is compact, then \( N(E) \) is open, and then apply Theorem 3.4.

**BIBLIOGRAPHY**


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