SOME INTEGRAL FORMULAS FOR CLOSED HYPERSURFACES

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Dedicated to the author's mother for her 70th birthday

Introduction. Let \( V^n \) be an orientable hypersurface twice differentially imbedded in a Euclidean space \( E^{n+1} \) of \( n + 1 \geq 3 \) dimensions, and let \( \kappa_1, \ldots, \kappa_n \) be the \( n \) principal curvatures at a point \( P \) of \( V^n \). The \( r \)-th mean curvature \( M_r \) of \( V^n \) at the point \( P \) is defined to be the \( r \)-th elementary symmetric function of \( \kappa_1, \ldots, \kappa_n \) divided by the number of terms, that is,

\[
(0.1) \quad \binom{n}{r} M_r = \sum \kappa_1 \ldots \kappa_r, \quad r = 1, \ldots, n.
\]

It is convenient to define \( M_0 = 1 \). Let \( p = p(P) \) denote the oriented distance from a fixed point \( O \) in \( E^{n+1} \) to the tangent hyperplane of \( V^n \) at \( P \), and let \( dA \) be the area element of \( V^n \) at \( P \). The purpose of this paper is first to show that for an orientable hypersurface \( V^n \) with a closed boundary \( V^{n-1} \) of dimension \( n-1 \) the integrals

\[
\int_{V^n} (M_{r+1}p + M_r)dA, \quad r = 0, \ldots, n-1,
\]

can be expressed as integrals over the boundary \( V^{n-1} \). These relations, which have been obtained by W. Scherrer [5] for \( n = 2 \), are then used to prove the following three theorems concerning closed hypersurfaces.

Theorem 1. Let \( V^n \) be a closed orientable hypersurface twice differentially imbedded in a Euclidean space \( E^{n+1} \) of \( n + 1 \geq 3 \) dimensions, then

\[
(0.2) \quad \int_{V^n} M_{r+1}p \, dA + \int_{V^n} M_r \, dA = 0, \quad r = 0, \ldots, n-1.
\]

For convex hypersurfaces, these formulas have been obtained by H. Minkowski for \( n = 2 \) and by T. Kubota for a general \( n \) (for references see [1, p. 64]).

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Theorem 2. Let $V^n$ satisfy the same conditions as in Theorem 1. Suppose that there exist a point $O$ in $E^{n+1}$ and an integer $s$, $1 \leq s \leq n-1$, such that $M_s > 0$ and either $p \leq -M_{s-1}/M_s$ or $p \geq -M_{s-1}/M_s$ at all points of $V^n$. Then $V^n$ is a hypersphere.

In the case where the hypersurface $V^n$ is convex, $s = 1$, and the equality holds in the last condition, this theorem has been obtained by K.-P. Grotemeyer [2] for $n = 2$ and by W. Süss [6] for a general $n$. Grotemeyer and Süss have also shown that a convex hypersurface satisfying a condition of the form $(-p)^s = 1/M_s$ is a hypersphere. It may be mentioned that this result can also be obtained for a more general class of hypersurfaces by using Theorem 1 and the method of Süss.

Theorem 3. Let $V^n$ satisfy the same conditions as in Theorem 1. Suppose that there exist a point $O$ in $E^{n+1}$ and an integer $s$, $1 \leq s \leq n$, such that at all points of $V^n$ the function $p$ is of the same sign, $M_i > 0$ for $i = 1, \ldots, s$, and $M_s$ is constant. Then $V^n$ is a hypersphere.

In the case $n = 2$, Theorem 3 reduces to the known results that a closed surface with constant Gaussian curvature is a sphere, and that a closed surface with constant mean curvature is necessarily a sphere if there exists a point which is on the same side of all tangent planes of the surface. For convex hypersurfaces of arbitrary dimensions the theorem is due to W. Süss (for references see [1, p. 118]). The proof of Theorem 3 is similar to that of Süss.

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1. Preliminaries. In a Euclidean space $E^{n+1}$ of dimension $n+1 \geq 3$ let us consider a fixed orthogonal frame $O \mathfrak{y}_1 \ldots \mathfrak{y}_{n+1}$ with a point $O$ as the origin. With respect to this orthogonal frame we define the vector product of $n$ vectors $A_1, \ldots, A_n$ in $E^{n+1}$ to be the vector $A_{n+1}$, denoted by $A_1 \times \ldots \times A_n$, satisfying the following conditions:

(a) the vector $A_{n+1}$ is normal to the $n$-dimensional space determined by the vectors $A_1, \ldots, A_n$,

(b) the magnitude of the vector $A_{n+1}$ is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \ldots, A_n$,

(c) the two frames $OA_1 \ldots A_n A_{n+1}$ and $O \mathfrak{y}_1 \ldots \mathfrak{y}_{n+1}$ have the same orientation.

Let $\sigma$ be a permutation on the $n$ numbers $1, \ldots, n$, then

\begin{equation}
A_{\sigma(1)} \times \ldots \times A_{\sigma(n)} = (\text{sgn} \sigma) A_1 \times \ldots \times A_n,
\end{equation}
where \(\text{sgn} \sigma = \pm 1\) according as the permutation \(\sigma\) is even or odd. Let \(i_1, \ldots, i_{n+1}\) be the unit vectors from the origin \(O\) in the directions of the vectors \(\mathcal{Y}_1, \ldots, \mathcal{Y}_{n+1}\) and let \(A^j_x, j = 1, \ldots, n+1\), be the components of the vector \(A_x, x = 1, \ldots, n\), with respect to the frame \(O \mathcal{Y}_1 \ldots \mathcal{Y}_{n+1}\), then the scalar product of any two vectors \(A_x\) and \(A_\beta\) and the vector product of \(n\) vectors \(A_1, \ldots, A_n\) are, respectively,

\[
A_x \cdot A_\beta = \sum_i A^i_x A^i_\beta,
\]

\[
A_1 \times A_2 \times \ldots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \ldots & i_{n+1} \\
A^1_1 & A^2_1 & \ldots & A^{n+1}_1 \\
\vdots & \vdots & \ddots & \vdots \\
A^1_n & A^2_n & \ldots & A^{n+1}_n \end{vmatrix}.
\]

If \(A^j_x\) are differentiable functions of \(n\) variables \(x^1, \ldots, x^n\), then by equation (1.3) and the differentiation of determinants,

\[
\frac{\partial}{\partial x^a} (A_1 \times \ldots \times A_n) = \sum_{\beta=1}^{n} \left( A_1 \times \ldots \times A_{\beta-1} \times \frac{\partial A_\beta^a}{\partial x^a} \times A_{\beta+1} \times \ldots \times A_n \right).
\]

Now we consider a hypersurface \(V^n\) twice differentiably imbedded in \(E^{n+1}\). Let \((y^1, \ldots, y^{n+1})\) be the coordinates of a point \(P\) in \(E^{n+1}\) with respect to the orthogonal frame \(O \mathcal{Y}_1 \ldots \mathcal{Y}_{n+1}\). Then \(V^n\) can be given by the parametric equations

\[
y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n+1,
\]

or the vector equation

\[
Y = F(x^1, \ldots, x^n),
\]

where \(y^i\) and \(f^i\) are respectively the components of the two vectors \(Y\) and \(F\), the parameters \(x^1, \ldots, x^n\) take values in a simply connected domain \(D\) of the \(n\)-dimensional real number space, \(f^i(x^1, \ldots, x^n)\) are of the second class and the Jacobian matrix \(\|\partial y^i/\partial x^a\|\) is of rank \(n\) at all points of \(D\). (See, for instance, also for the remainder of this section, \([7, \text{Chap. IX}])\) For quantities of the \(V^n\), tensor notation with Greek indices will be used. In particular, the summation convention is adopted for these indices. If we denote the vector \(\partial Y/\partial x^a\) by \(Y_\alpha\) for \(\alpha = 1, \ldots, n\), then the first fundamental form of \(V^n\) at a point \(P\) is

\[
ds^2 = g_{\alpha \beta} \, dx^\alpha \, dx^\beta, \quad g_{\alpha \beta} = Y_\alpha \cdot Y_\beta,
\]

where the matrix \(\|g_{\alpha \beta}\|\) is positive definite and thus, the determinant
Let $N$ be the unit normal vector at a point $P$ of $V^n$ and $N_\alpha$ the vector $\partial N/\partial x^\alpha$, then

\begin{equation}
N_\alpha = - b_{\alpha\beta} g^{\beta\gamma} Y_\gamma ,
\end{equation}

where

\begin{equation}
b_{\alpha\beta} = b_{\beta\alpha} = - N_\alpha \cdot Y_\beta
\end{equation}

are the coefficients of the second fundamental form of $V^n$ and $g^{\beta\gamma}$ denotes the cofactor of $g_{\beta\gamma}$ in $g$ divided by $g$ so that

\begin{equation}
g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma ,
\end{equation}

$\delta^\alpha_\gamma$ being the Kronecker deltas. The $n$ principal curvatures $\kappa_1, \ldots, \kappa_n$ of $V^n$ at $P$ are the roots of the determinant equation

\begin{equation}
|b_{\alpha\beta} - \kappa g_{\alpha\beta}| = 0 .
\end{equation}

From equations (0.1) and (1.12) follow immediately

\begin{equation}M_n = b/g, \quad n M_1 = b_{\beta\alpha} g^{\alpha\beta} , \quad n M_{n-1} = g_{\alpha\beta} B^{\alpha\beta}/g ,
\end{equation}

where

\begin{equation}b = |b_{\alpha\beta}| ,
\end{equation}

and $B^{\alpha\beta}$ is the cofactor of $b_{\alpha\beta}$ in $b$.

The area element of $V^n$ at $P$ is given by

\begin{equation}dA = g^{1/2} dx^1 \ldots dx^n .
\end{equation}

Now we choose the direction of the unit normal vector $N$ in such a way that the two frames $PY_1 \ldots Y_n N$ and $O \emptyset_1 \ldots \emptyset_{n+1}$ have the same orientation. Then from equations (1.3) and (1.15) it follows that

\begin{equation}g^{1/2} N = Y_1 \times \ldots \times Y_n ,
\end{equation}

\begin{equation}|Y_1, \ldots, Y_n, N| = g^{1/2} .
\end{equation}

2. Proof of the formula (0.2) for $r = 0$. At first, we observe that the vector $Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n$ is perpendicular to the normal vector $N$ and can therefore be written in the form

\begin{equation}Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n = a^{\alpha\beta} Y_\beta .
\end{equation}

Taking the scalar products of the both sides of equations (2.1) with the vector $Y_\gamma$ and making use of equations (1.1), (1.3), (1.7), (1.16), we obtain

\begin{equation}a^{\alpha\beta} g_{\beta\gamma} = - g^{1/2} \delta^\alpha_\gamma , \quad \alpha, \gamma = 1, \ldots, n .
\end{equation}
Solving equations (2.2) for $a^{\alpha\beta}$ for each fixed $\alpha$ and substituting the results in equations (2.1), we are led to

\begin{equation}
Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n = -g^{1/2} g^{\alpha\beta} Y_{\beta}.
\end{equation}

Making use of equations (1.4), (1.9), (1.13) and (1.16), it is easily seen that

\begin{equation}
\sum_{\alpha=1}^{n} \frac{\partial}{\partial x^{\alpha}} (Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n) = \sum_{\alpha=1}^{n} Y_1 \times \ldots \times Y_{\alpha-1} \times N_{\alpha} \times Y_{\alpha+1} \times \ldots \times Y_n = -ng^{1/2} M_1 N.
\end{equation}

Thus, from equations (2.3) and (2.4),

\begin{equation}
g^{1/2} M_1 N = \frac{\partial}{\partial x^{\alpha}} (g^{1/2} g^{\alpha\beta} Y_{\beta}).
\end{equation}

Taking the scalar products of the both sides of equation (2.5) with the vector $Y$, we obtain in consequence of the relations (1.7) and (1.11)

\begin{equation}
n M_1 p g^{1/2} = \frac{\partial}{\partial x^{\alpha}} (g^{1/2} g^{\alpha\beta} \eta_{\beta}) - ng^{1/2},
\end{equation}

where we have put

\begin{equation}
p = Y \cdot N, \quad \eta_{\alpha} = Y \cdot Y_{\alpha}.
\end{equation}

Now let us consider a hypersurface $V^n$ having a closed boundary $V^{n-1}$ and twice differentiably imbedded in a Euclidean space $E^{n+1}$ of $n+1 \geq 3$ dimensions. Integrating equation (2.6) with respect to $x^1, \ldots, x^n$ over this hypersurface $V^n$ and applying the general Green’s theorem (cf., for instance, [4, pp. 75–76]) to the first term on the right side of equation (2.6), we then obtain

\begin{equation}
\int_{V^n} M_1 p dA + A = n^{-1} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} g^{1/2} g^{\alpha\beta} \eta_{\beta} dx^1 \ldots dx^{\alpha-1} dx^{\alpha+1} \ldots dx^n.
\end{equation}

In particular, when $V^n$ is closed and orientable the integral on the right side of equation (2.8) drops out and hence the formula (0.2) for $r = 0$ follows.

3. Proof of the formula (0.2) for a general $r$. In this section we shall use the formula (2.8) to derive an analogous formula for a general $r$. To this end, in $E^{n+1}$ we first consider a hypersurface $\overline{V}^n$ parallel to a hypersurface $V^n$ with a closed boundary $V^{n-1}$ so that $\overline{V}^n$ and $V^n$ have
the same normals. It is evident that the vector equation of $\overline{V}^n$ can be written in the form

$$\overline{Y} = Y - tN,$$

where $t$ is a real parameter. From equations (3.1), $N \cdot N = 1$ and $N \cdot \overline{Y}_\alpha = 0$, it follows immediately that $\partial t/\partial \alpha = 0$ and therefore that $t$ is constant. Making use of equations (1.7), (1.9), (1.10) and their analogous ones for $\overline{V}^n$ we obtain the coefficients of the first and the second fundamental forms of $\overline{V}^n$:

$$\bar{g}_{\alpha \beta} = g_{\alpha \beta} + 2b_{\alpha \beta}t + b_{\alpha \sigma}b_{\beta \sigma}g^{\sigma \sigma}t^2 = (g_{\alpha \sigma} + b_{\alpha \sigma}t)(\delta^{\sigma}_\beta + b_{\beta \sigma}g^{\sigma \sigma}t),$$

$$\varrho_{\alpha \beta} = b_{\alpha \beta} + b_{\alpha \sigma}b_{\beta \sigma}g^{\sigma \sigma}t = b_{\alpha \beta}(\delta^{\alpha}_\beta + b_{\beta \sigma}g^{\sigma \sigma}t),$$

from which it follows easily by an elementary calculation that

$$\bar{g} = gA^2,$$

$$\varrho = gA^2,$$

$$|(\overline{R} - t)_{\alpha \beta} - g_{\alpha \beta}| = A,$$

where $\varrho$ and $b$ are defined by equations similar to (1.8), (1.14), and

$$A = |\delta^\alpha_\beta + b_{\alpha \sigma}g^{\sigma \beta}t|,$$

$$\overline{R}_i = \frac{1}{\bar{\kappa}_i}, \quad i = 1, \ldots, n,$$

$\bar{\kappa}_i$ being the principal curvatures of $\overline{V}^n$. In consequence of equations (3.4), (3.5), (3.6) and (1.12), (1.13), (1.15) together with their analogues for $\overline{V}^n$, we have

$$\overline{M}_n d\overline{A} = M_n dA,$$

$$\overline{R}_i = R_i + t,$$

where $d\overline{A}$ is the area element of $\overline{V}^n$ and $R_i = 1/\bar{\kappa}_i$. Moreover, let $\bar{g}^\alpha_\beta$ be the cofactor of $\bar{g}_{\alpha \beta}$ in $\bar{g}$ divided by $\bar{g}$, then from equations (2.7), (3.1), (3.2) and (3.7) we obtain

$$\bar{\eta}_\beta = \eta_\beta + t\beta_\beta g^{\gamma \gamma} \eta_\gamma = \eta_\gamma (\delta^\gamma_\beta + t\beta_\beta g^{\gamma \gamma}),$$

$$\bar{g} \bar{g}^{\alpha \beta} \bar{\eta}_\beta = \Phi^\alpha A,$$

where $\bar{\eta}_\beta = \overline{Y} \cdot \overline{Y}_\beta$ and
\begin{equation}
\Phi^x = \begin{vmatrix}
g_{11} + tb_{11} & g_{12} + tb_{12} & \cdots & g_{1n} + tb_{1n} \\
g_{\alpha-1,1} + tb_{\alpha-1,1} & g_{\alpha-1,2} + tb_{\alpha-1,2} & \cdots & g_{\alpha-1,n} + tb_{\alpha-1,n} \\
\eta_1 & \eta_2 & \cdots & \eta_n \\
g_{\alpha+1,1} + tb_{\alpha+1,1} & g_{\alpha+1,2} + tb_{\alpha+1,2} & \cdots & g_{\alpha+1,n} + tb_{\alpha+1,n} \\
g_{n1} + tb_{n1} & g_{n2} + tb_{n2} & \cdots & g_{nn} + tb_{nn}
\end{vmatrix}.
\end{equation}

Now let
\begin{equation}
\Phi^x = \sum_{r=0}^{n-1} \binom{n-1}{r} \Theta^x_r \tau^r,
\end{equation}
then it is obvious that
\begin{equation}
\Theta^x_0 = g g^{\alpha \beta} \eta_\beta, \quad \Theta^x_{n-1} = B^{\alpha \beta} \eta_\beta.
\end{equation}

By means of equations (0.1) and (3.8), equation (2.8) for $\overline{V}^n$ can be written as
\begin{equation}
\int_{\overline{V}^n} \rho (\Sigma R_1 \overline{R}_2 \ldots \overline{R}_{n-1}) M_n \, d\overline{A} + n \int_{\overline{V}^n} \overline{R}_1 \overline{R}_2 \ldots \overline{R}_n \, M_n \, d\overline{A}
= \int_{\overline{V}^{n-1}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} g^{1/2} \overline{g}^{\alpha \beta} \overline{\eta}_\beta \, dx^1 \ldots dx^{\alpha-1} \, dx^{\alpha+1} \ldots dx^n,
\end{equation}
where $\rho = \overline{Y} \cdot N = p-t$ and $\overline{V}^{n-1}$ is the boundary of $\overline{V}^n$. Substitution of equations (3.5), (3.9), (3.10), (3.12) and (3.14) in equation (3.16) yields immediately
\begin{equation}
\int_{\overline{V}^n} (p-t) \sum_{i=0}^{n-1} (n-i)(\Sigma R_1 \ldots R_i) t^{n-i-1} \, M_n \, d\overline{A} +
\int_{\overline{V}^n} \sum_{i=0}^{n} (\Sigma R_1 \ldots R_i) t^{n-i} \, M_n \, d\overline{A}
= \int_{\overline{V}^{n-1}} \sum_{\alpha=1}^{n} \sum_{r=0}^{n-1} (-1)^{\alpha-1} \binom{n-1}{r} g^{1/2} \Theta^x_r \tau^r \, dx^1 \ldots dx^{\alpha-1} \, dx^{\alpha+1} \ldots dx^n,
\end{equation}
which is an identity in $t$. Hence, by equating the coefficients of $t^r$ on the both sides of equation (3.17) and using (0.1), we arrive at the generalization of the formula (2.8) mentioned in the introduction:
\[(3.18) \quad \int_{V^n} M_{r+1} p \, dA \, + \int_{V^n} M_r \, dA \\
= n^{-1} \int_{V_{n-1}} \sum_{s=1}^{n} (-1)^{s-1} g^{-1/2} \Theta_s^r \, dx^1 \ldots \, dx^{s-1} \, dx^{s+1} \ldots \, dx^n , \quad r = 0, \ldots, n-1 ,\]

from which follow immediately the formulas (0.2) when \(V^n\) is closed and orientable.

**4. Proofs of Theorems 2 and 3.** In order to prove Theorem 2 we first observe that because of \(M_s > 0\) the assumptions \(p \leq -M_{s-1}/M_s\) and \(p \geq -M_{s-1}/M_s\) are respectively equivalent to \(M_s p + M_{s-1} \leq 0\) and \(M_s p + M_{s-1} \geq 0\). From (0.2) for \(r = s-1\) we have

\[\int_{V^n} (M_s p + M_{s-1}) \, dA = 0 .\]

Hence, either assumption implies \(p = -M_{s-1}/M_s\). Substituting this in (0.2) for \(r = s\), we obtain

\[(4.1) \quad \int_{V^n} (M_s^2 - M_{s-1} M_{s+1})/M_s \, dA = 0 .\]

Since \((\binom{n}{i}) M_i\) is the \(i\)-th elementary symmetric function of the real numbers \(\xi_1, \ldots, \xi_n\), we have the inequalities

\[(4.2) \quad M_i^2 - M_{i-1} M_{i+1} \geq 0, \quad i = 1, \ldots, n-1 ,\]

and equality in (4.2) for any value of \(i\) implies \(\xi_1 = \ldots = \xi_n\) (cf. [3, pp. 52, 104]). From (4.1) it follows therefore that \(\xi_1 = \ldots = \xi_n\) at all points of \(V^n\). It is well known that this implies that \(V^n\) is a hypersphere, and hence Theorem 2 is proved.

If \(M_{i-1} > 0\) and \(M_i > 0\), the inequality (4.2) may be written as

\[(4.3) \quad M_i / M_{i-1} \geq M_{i+1} / M_i .\]

Let the assumptions of Theorem 3 be satisfied for some \(s < n\). Then the inequality (4.3) holds for \(i = 1, \ldots, s\). In particular, we have \(M_1 / M_0 \geq M_{s+1} / M_s\) or

\[(4.4) \quad M_1 M_s \geq M_{s+1} ,\]

and the equality implies \(\xi_1 = \ldots = \xi_n\). Since \(M_1 > 0\) and it is assumed that \(p\) has the same sign at all points of \(V^n\), we must have \(p < 0\)
because of the formula (0.2) for \( r = 0 \). Multiplying the both sides of the inequality (4.4) by \( p \), integrating over \( V^n \), and applying the formula (0.2) for \( r = 0 \) and \( r = s \), we obtain
\[
-M_s \int_{V^n} dA = M_s \int_{V^n} M_1 p \, dA \leq \int_{V^n} M_{s+1} p \, dA = -M_s \int_{V^n} dA,
\]
since \( M_s \) is constant. Consequently, equality must hold in (4.4) at all points of \( V^n \), and hence Theorem 3 for \( s < n \) follows.

In the remaining case of Theorem 3, where \( s = n \), the assumptions imply
\[
(4.5) \quad M_i > 0, \quad i = 1, \ldots, n.
\]
It is known that from the inequalities (4.2) and (4.5) it follows that
\[
(4.6) \quad M_1 \geq M_2^{1/2} \geq \ldots \geq M_{n-1}^{1/(n-1)} \geq M_n^{1/n},
\]
and equality at any stage in (4.6) implies \( \kappa_1 = \ldots = \kappa_n \) (cf. [3, p. 52]). Now put \( M_n = c^n \), where \( c \) is a positive constant. Then we obtain on one hand, by means of the formula (0.2) for \( r = n-1 \) and the inequalities (4.6),
\[
\int_{V^n} M_n p \, dA = -\int_{V^n} M_{n-1} \, dA \leq -c^{n-1} \int_{V^n} dA,
\]
and on the other hand, by means of \( p < 0 \), the inequalities (4.6) and the formula (0.2) for \( r = 0 \),
\[
\int_{V^n} M_n p \, dA = c^{n-1} \int_{V^n} M_n^{1/n} p \, dA \geq c^{n-1} \int_{V^n} M_1 p \, dA = -c^{n-1} \int_{V^n} dA.
\]
Thus \( M_n^{1/n} = M_1 \) and again we have \( \kappa_1 = \ldots = \kappa_n = c \) at all points of \( V^n \).

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