NOTE ON A GENERALIZATION OF A THEOREM OF BOGOLIOUOBOFF

ERLING FØLNER

The aim of the present note is to show that some of the theorems in a recent paper by the author [4] can be improved in a simple way. It will be assumed that the reader is familiar with this paper.

In Theorem 1 of [4, p. 6] the assumption that $E$ is relatively dense with respect to $k$ elements can be replaced by the assumption that $\bar{m} E > 0$. In the conclusion of the theorem the statement $q \leq k^2$ should then be replaced by $q \leq (\bar{m} E)^{-2}$.

When $E$ is relatively dense with respect to $k$ elements, it follows that $\bar{m} E \geq m E \geq k^{-1} > 0$ and hence $k^2 \geq (\bar{m} E)^{-2} \geq (m E)^{-2}$. Thus the new form of Theorem 1 is actually stronger than the original one.

As mentioned in [4, p. 6] Bogoliouboff [2] proved Theorem 1 (in a somewhat different form) for the case where $G$ is the discrete additive group of all integers. From this case he passed to the case where $G$ is the additive group of all real numbers with the usual topology. In the proof for the group of integers Bogoliouboff uses only that the upper Besicovitch mean measure $\bar{m}_B E$ of $E$ is positive. For this group our $\bar{m} E$ is easily seen to be the upper Weyl mean measure $\bar{m}_W E$ of $E$, and the condition $\bar{m}_W E > 0$ is weaker than the condition $\bar{m}_B E > 0$ since $\bar{m}_W E \geq \bar{m}_B E$. But a simple change in Bogoliouboff’s proof will make it work with the assumption $\bar{m}_W E > 0$.

Stronger forms of Theorem 2 and Theorem 3 of [4, p. 7] and Corollary 1 (as regards the sufficient condition) and Corollary 2 (as regards the necessary condition) of [4, p. 6] can be obtained by replacing the relatively dense sets $E$ which occur in these theorems by sets $E$ with $\bar{m} E > 0$.

We shall modify the proof of Theorem 1 so as to obtain a proof of the stronger form of Theorem 1. Instead of the theorem of Banach cited in [4, p. 8] we shall use a stronger form of it which is also due to Banach [1, pp. 27–28].

Banach’s Theorem. Let $\bar{M} f$ be a real functional defined on a real linear space $L$ and satisfying (4) and (5) in [4, p. 8]. Let further $M f$ be a

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linear functional on a linear subspace of \( L \) and satisfying the inequality \( Mf \leq \overline{M}f \). Then \( Mf \) can be extended to a linear functional on \( L \) which satisfies the same inequality on the whole of \( L \).

As a simple consequence of this theorem the functional \( Mf \) chosen in [4, p. 8] may be chosen so that \( Mj = \overline{M}j \) for the special function \( j(x) \) introduced in [4, p. 9]. In particular \( Mj \geq \tilde{m}E > 0 \).

The proof of the stronger form of Theorem 1 is now obtained from the original proof of Theorem 1 by replacing the relation (1) in [4, p. 10] and the argument leading to it by the relation

\[(1') \quad \mathfrak{M} \mu \geq (Mj)^2\]

and the following proof of (1').

Let \( j_1(x) = j(x) - Mj \). Then

\[
\mu(x) = \mathcal{M} \{ (j(t)+Mj)(j_1(t+x)+Mj) \}
\]

\[
= \mathcal{M} \{ j_1(t+j_1(t+x)) \} + (Mj)^2.
\]

It was proved in [4, p. 9] that \( \mu(x) \) is a positive definite function. It follows in the same way that

\[
\mu_1(x) = \mathcal{M} \{ j_1(t)+j_1(t+x) \}
\]

is a positive definite function. We conclude, applying, for instance, (3), (4) in [4, p. 10] to \( \mu_1 \) instead of \( \mu \), that \( \mathfrak{R} \mu_1 \geq 0 \). Thus (1') is obtained by taking mean values in the above relation for \( \mu(x) \).

The stronger forms of the other theorems are now proved in the same way as before. Only for the stronger form of Theorem 2 a minor modification is needed in order to prove that \( f(x) \) is bounded.

We shall use the fact that when \( \tilde{m}E > 0 \), the set \( E - E \) is relatively dense. This has been proved directly in [3, p. 61], but it is also a simple corollary of the present stronger form of Theorem 3.

Since \( E \) consists of \( \epsilon_0 \)-translation elements of \( f(x) \), the set \( E - E \) consists of \( 2\epsilon_0 \)-translation elements of \( f(x) \). The boundedness of \( f(x) \) now follows as in [4, p. 13] if \( E \) is replaced by \( E - E \) and \( \epsilon_0 \) by \( 2\epsilon_0 \).

REFERENCES


THE TECHNICAL UNIVERSITY OF DENMARK, COPENHAGEN