MATH. SCAND. 2 (1954), 19–28

ON CONVEX CONES

LENNART SANDGREN

Introduction. In two well-known theorems concerning convex sets in euclidean n-space, viz. Carathéodory's theorem (stating that every point of the convex hull of a point set M is a non-negative linear combination of $n+1$ points of M) and Helly's theorem (stating that arbitrarily many compact convex sets have a common point if any $n+1$ of them have a common point) the dimension of the space enters in a manner which suggests searching for a reciprocity between these theorems. In this paper, by means of the notion of the dual or polar of a convex cone, such a reciprocity is shown to exist. Also, it turns out to be suitable to deduce certain other theorems of the Helly type from theorems of the Carathéodory type.

In the first three sections definitions and basic properties of convex cones and their duals are given. Most of these properties are well known (cf. Steinitz [3]). In Section 4, as a simple application, an expression due to F. Riesz for the support function of the intersection of convex bodies is derived. The above mentioned proof of Helly's theorem is given in Section 5. In connection with this proof the question arises: Given a set of closed convex cones; which assumptions guarantee the existence of a hyperplane such that each of the cones is contained in one of the closed half spaces bounded by this hyperplane. In Section 6 this question is answered, a sufficient condition being that the convex hull of any n of the cones is not the whole space. In Section 7 the dual of this theorem is stated. It turns out to be essentially equivalent to a generalization of Helly's theorem due to A. Horn [2]. Thus, the latter theorem is proved in a new manner.

1. Definitions and notations. Let E^n denote the n-dimensional euclidean space with points

$$O = (0, \ldots, 0), \quad X = (x_1, \ldots, x_n), \quad Y = (y_1, \ldots, y_n), \ldots$$

and the inner product

Received October 1, 1953.

The author is indebted to Professors M. Riesz and W. Fenchel for many stimulating discussions concerning various points of this paper.

$$(X,\ Y) = x_1y_1 + \ \dots\ + x_ny_n\,.$$

For the sake of brevity, an $(n-1)$-dimensional subspace or hyperplane, that is the set of points X satisfying an equation of the form $(X,\ Y) = \text{const.}$ with $Y \neq O$ fixed, is called a *plane*. The set of points X satisfying $(X,\ Y) \leqq \text{const.}$ $\big((X,\ Y) < \text{const.}\big)$ is called a *closed* (*open*) *half space* and Y its *outer normal*.

A plane is said to *divide* a point set M if each of the open half spaces bounded by the plane contains points of M. A plane is said to *separate* two point sets if each of the two closed half spaces bounded by the plane contains one of the sets. Observe that a subset of a plane is not divided by this plane, and that two subsets of a plane are separated by this plane.

Let M be an arbitrary point set. The intersection of all convex sets which contain M is called the *convex hull* of M and is denoted by $\mathcal{H}M$. The *closed convex hull* or *convex closure* of M, denoted by $\mathcal{C}M$, is the intersection of all closed convex sets containing M. It is well known that if M is contained in some closed half space, then $\mathcal{C}M$ is the intersection of all such half spaces. If M is contained in no half space, then $\mathcal{C}M = \mathcal{H}M = E^n$.

A set M is called a *cone* with vertex O if $X \in M$ implies $\lambda X = (\lambda x_1,\ \dots,\ \lambda x_n) \in M$ for every $\lambda \geqq 0$. Note that the point O is a cone. In this paper all cones will have O as vertex. As usual, the opposite of a cone M, that is, the set of all points $-X$, $X \in M$, will be denoted by $-M$. A *ray* OX where $X \neq O$, that is, the set of points λX, $\lambda \geqq 0$, is an example of a cone. Every cone except O is a union of rays. A cone M is *convex* if and only if $X \in M$, $Y \in M$ imply $X+Y = (x_1+y_1,\ \dots,\ x_n+y_n) \in M$.

We are dealing mainly with closed convex cones. According to our definition both O and E^n are closed convex cones. In some of the following theorems these degenerate cones play an important role. A closed convex cone is either the whole space E^n or it is contained in some closed half space $(X,\ Y) \leqq 0$ and is then the intersection of all such half spaces. In the latter case there exist supporting planes, namely the planes bounding these half spaces.

Let C be a convex cone and denote by d the maximum number of linearly independent vectors OX contained in C. Then C and the smallest subspace containing C have dimension d. Considered as a subset of this subspace, C has interior points, and if C is not identical with the subspace, it has boundary points also. These will be called the *relative interior points* and the *relative boundary points* of C, respectively. If $X \neq O$ is a relative boundary point of C, the ray OX is called a *relative boundary ray*.

We recall the well-known separation theorem:

THEOREM 1.1. *If the relative interiors of two convex cones have no points in common, then there exists a plane which separates the cones.*

Obviously there are convex cones containing entire subspaces through O. The largest subspace through O contained in a convex cone C is called the *lineality space* and its dimension l is called the *lineality* of C. We shall apply the following theorem concerning these notions.

THEOREM 1.2. *If a convex cone C contains points $X_1, \ldots, X_k$ satisfying a linear relation*

$$\mu_1 X_1 + \ldots + \mu_k X_k = O$$

with positive coefficients $\mu_\varkappa$, then the lineality space of C contains the subspace spanned by the points $O, X_1, \ldots, X_k$ and hence, the lineality of C is greater than or equal to the maximal number of linearly independent points among $X_1, \ldots, X_k$.

This is seen in the following way. Let $X = \lambda_1 X_1 + \ldots + \lambda_k X_k$ be an arbitrary point of this subspace. Choose a number $\varrho > 0$ such that $\lambda_\varkappa + \varrho\mu_\varkappa \geqq 0$ for $\varkappa = 1, \ldots, k$. Then

$$X = (\lambda_1 + \varrho\mu_1)X_1 + \ldots + (\lambda_k + \varrho\mu_k)X_k$$

is a representation of X as a sum of points belonging to C and, hence, $X \in C$.

2. Dual cones. Let C be a closed convex cone. The set

$$C^* = \{X \mid (X, Y) \leqq 0 \text{ for all } Y \in C\}$$

is also a closed convex cone called the *dual* or *polar cone* of C. If $C \neq O$, the dual cone C^* is the intersection of all half spaces $(X, Y) \leqq 0$, the outer normals of which are rays in C. In other words, C^* is the set of rays making an angle $\geqq \frac{1}{2}\pi$ with all rays belonging to C. Further, $O^* = E^n$ and $E^{n*} = O$. From the fact that the cone C is the intersection of all half spaces containing it, it follows easily that

$$C^{**} = C\,.$$

Between the dimension of a convex cone and the lineality of its dual cone we have the following relation.

A closed convex cone C has lineality l if and only if the dual cone C^* is $(n-l)$-dimensional.

From the definition of the dual cone it follows immediately that if C_1

and C_2 are closed convex cones and $C_1 \subseteq C_2$, then $C_1^* \supseteq C_2^*$. Using this, we can easily prove the following Theorem 2, which gives the relation between the intersection of convex cones and the convex hull of the corresponding dual cones.

Throughout this paper $\{C_A\}$ will denote a set of convex cones, the index A running through any finite or infinite set.

THEOREM 2. *If $\{C_A\}$ is a set of closed convex cones, then*

$$(\bigcap C_A)^* = \mathscr{C} \bigcup C_A^* .$$

PROOF. Evidently, $\bigcap C_A \subseteq C_B$ and thus, $(\bigcap C_A)^* \supseteq C_B^*$ holds for every B. Hence, $(\bigcap C_A)^* \supseteq \bigcup C_B^*$ and since $(\bigcap C_A)^*$ is closed and convex, we have $(\bigcap C_A)^* \supseteq \mathscr{C} \bigcup C_A^*$. Conversely, $C_B^* \subseteq \mathscr{C} \bigcup C_A^*$ and thus, $C_B \supseteq (\mathscr{C} \bigcup C_A^*)^*$. Hence, $\bigcap C_B \supseteq (\mathscr{C} \bigcup C_A^*)^*$. Forming dual cones we obtain $(\bigcap C_A)^* \subseteq \mathscr{C} \bigcup C_A^*$.

From this theorem we have the important

COROLLARY. *If $\{C_A\}$ is a set of closed convex cones, then $\bigcap C_A = O$ if and only if $\mathscr{H} \bigcup C_A^* = E^n$.*

3. Sum representation of the convex hull of a cone. The following proposition is well known.

THEOREM 3.1. *Let M be an arbitrary cone. Then every point $X \in \mathscr{H} M$ has a representation $X = X_1 + \ldots + X_r$ where $X_j \in M$ and $X_1, \ldots, X_r$ are linearly independent, hence, $r \leqq n$.*

We formulate an immediate consequence, which will be used in the sequel.

COROLLARY. *Let $\{M_A\}$ be a set of cones. Suppose that there exists a ray OX such that the convex hull of any n of the cones M_A does not contain OX. Then the convex hull of all of the cones M_A does not contain OX. Hence, there is a half space containing all of the cones M_A.*

For, if $X \in \mathscr{H} \bigcup M_A$, there would exist a representation of X of the above form with each X_j contained in some M_A. Consequently, X would belong to the convex hull of at most $r \leqq n$ of the cones M_A.

Another consequence of Theorem 3.1 to be used in the sequel is

THEOREM 3.2. *Let M be an arbitrary cone. Further, let X be a point of $\mathscr{H} M$, and let Z be a point which is not interior to $\mathscr{H} M$. Then there exists, on the closed segment XZ, a point Y having a representation $Y = Y_1 + \ldots + Y_s$, where $Y_j \in M$ and $s \leqq n-1$.*

PROOF. Let $X = X_1 + \ldots + X_r$ be a representation of X according to Theorem 3.1 with $X_1, \ldots, X_r$ linearly independent. If $r \leqq n-1$, we may choose $Y = X$. If $r = n$, the point X is an interior point of the convex hull of the rays $OX_1, \ldots, OX_n$. This convex hull is a polyhedral convex cone P of dimension n, the interior points of which are interior points of $\mathcal{H}M$. Hence, Z cannot be interior to P and consequently, the segment XZ contains a boundary point Y of P. Obviously, Y has a representation with the desired property.

4. The support function of the intersection of convex bodies. Let m be an arbitrary set of a ν-dimensional euclidean space E^ν. Put $n = \nu + 1$ and consider E^ν as the plane $x_n = -1$ of E^n. To the set $m \subseteq E^\nu$ associate the cone $M \subseteq E^n$ consisting of all rays OX, $X \in m$.

Now consider a convex body $K \subseteq E^\nu$, that is, a closed bounded convex set. There is a simple relationship between the dual cone C^* of the cone C associated with K and the support function of K. When a point $X = (x_1, \ldots, x_\nu, -1) \in E^n$ is considered only in E^ν we denote it by $x = (x_1, \ldots, x_\nu)$. Obviously, the definition

$$C^* = \{Y \mid (X, Y) \leqq 0 \text{ for } X \in C\}$$

is equivalent to

$$C^* = \{Y \mid x_1 y_1 + \ldots + x_\nu y_\nu \leqq y_n \text{ for } x \in K\}.$$

Now

$$H(y) = \sup_{x \in K} (x_1 y_1 + \ldots + x_\nu y_\nu)$$

is the support function of K. Thus, C^* consists of the points $Y = (y_1, \ldots, y_n)$ satisfying $y_n \geqq H(y)$ and hence, the boundary of C^* has the equation $y_n = H(y)$; in other words, C^* represents the support function of K.

Let $\{K_A\}$ be a set of convex bodies in E^ν, and let $C_A \subseteq E^n$ denote the cone associated with K_A. Then the cone associated with the intersection $\bigcap K_A$ is $\bigcap C_A$. From Theorem 2 it follows therefore that the support function of $\bigcap K_A$ is represented by the cone $\mathscr{C} \bigcup C_A^*$, this cone being the whole space E^n if and only if $\bigcap C_A = O$, that is, if and only if $\bigcap K_A$ is empty. This together with Theorem 3.1 applied to $M = \bigcup C_A^*$ yields the following (unpublished) theorem of F. Riesz.

THEOREM 4. *Let $\{K_A\}$ be a set of convex bodies in E^ν, and let $H_A(x)$ denote the support function of K_A. Define for $x \in E^\nu$*

$$H(x) = \inf \left(H_{A_1}(x^{(1)}) + \ldots + H_{A_{\nu+1}}(x^{(\nu+1)}) \right),$$

where the infimum is taken over all sets of points $x^{(1)}, \ldots, x^{(\nu+1)}$ in E^ν for *which* $x^{(1)}+\ldots+x^{(\nu+1)}=x$ *and for all combinations* $A_1, \ldots, A_{\nu+1}$. *Then* $H(x)$ *is the support function of* $\bigcap K_A$ *if this intersection is not empty, and* $H(x)=-\infty$ *for all* x *if this intersection is empty.*

5. A proof of Helly's theorem. By means of Theorem 2 and the corollary to Theorem 3.1, we now give a simple proof of Helly's well-known theorem:

THEOREM 5. *Let* $\{K_A\}$ *be a set of closed convex sets in* E^ν *with the properties that any* $\nu+1$ *of the sets* K_A *have a common point and that some of the sets have a bounded (non-empty) intersection. Then all of the sets* K_A *have a common point.*

PROOF. As in the preceding section, let E^ν be the plane $x_n=-1$ of E^n, $n=\nu+1$, and associate with each set K_A the convex cone projecting it from O. This cone is closed if K_A is bounded. Denote by C_A this cone or its closure in case K_A is unbounded. Let $C_{A_1}, \ldots, C_{A_n}$ be n of these cones. According to the assumption there is a point X of E^ν for which $X \in C_{A_1} \cap \ldots \cap C_{A_n}$. With the notation $N=(0, \ldots, 0, -1)$ we have $(X, N)=1>0$. Thus, the definition of the dual cone and Theorem 2 give

$$N \notin (C_{A_1} \cap \ldots \cap C_{A_n})^* = \mathscr{C}(C_{A_1}^* \cup \ldots \cup C_{A_n}^*)\,.$$

Now, the corollary to Theorem 3.1 applied to the cones $M_A = C_A^*$ shows that $N \notin \mathscr{H} \bigcup C_A^*$. Consequently, $\mathscr{H} \bigcup C_A^*$ is not the whole space E^n and, by the corollary to Theorem 2, $\bigcap C_A$ is not O. Since there are sets K_A having a bounded intersection, $\bigcap C_A$ must have points in common with E^ν that is, all K_A have a common point.

6. A separation theorem for convex cones. In the corollary to Theorem 3.1 it was assumed that the convex hull of any n of the cones does not contain a fixed ray, and the conclusion was that there exists a plane having all of the cones on the same side. Now consider a set of closed convex cones C_A and weaken the assumption to this: the convex hull of any n of the cones is not the entire space E^n. In general it is then not true that all of the cones are in the same half space. As an example in E^2 take three rays forming equal angles with each other. However, we are going to show that there exists a plane which divides no C_A. In other words, we shall prove the following

THEOREM 6. *Let* $\{C_A\}$ *be a set of closed convex cones in* E^n *with the property that the convex hull of any* n *of the cones* C_A *is not the whole space* E^n. *Then there exists a plane which divides no cone* C_A.

If all of the cones C_A are n-dimensional, this plane may be chosen such as to have n preassigned cones C_A on the same side.

Observe that the second statement does not necessarily hold if not all of the C_A are n-dimensional. As an example in E^2 choose for C_1 the quadrant $x_1 \geqq 0$, $x_2 \geqq 0$, for C_2 the negative x_1-axis, and for C_3 the entire x_2-axis. Then the latter is the only line satisfying the requirement of the theorem. Nevertheless it has not C_1 and C_2 on the same side.

PROOF. First assume that the number of cones C_A is finite.

Choose n arbitrary cones C_A and denote them by $C_1, \ldots, C_n$. To these cones add a maximal number of the cones C_A, say $C_{n+1}, \ldots, C_k$, such that $\mathcal{H}(C_1 \cup \ldots \cup C_k)$ is not the whole space and thus, contained in some half space. Define

$$\Gamma = \mathcal{H}(C_1 \cup \ldots \cup C_k) \, .$$

Let C_j, $j > k$, be one of the remaining cones, if any. Then $-\Gamma$ and C_j cannot be separated by a plane; for, a plane separating $-\Gamma$ and C_j would have the $k+1$ cones $C_1, \ldots, C_k, C_j$ on the same side, in contradiction to the maximum property of k. From Theorem 1.1 it therefore follows that C_j and $-\Gamma$ have points in common which are relative interior to both cones. Since Γ and $-\Gamma$ are separated by every supporting plane of Γ, it is sufficient to show that C_j is contained in the closure $-\bar{\Gamma}$ of $-\Gamma$ for all $j > k$.

There is thus a point, call it X, which is relative interior to both C_j and $-\Gamma$. Suppose C_j is not contained in $-\bar{\Gamma}$. Then there is a point Z relative interior to C_j but not contained in $-\Gamma$. Let Y be that point, or one of those points, of the segment XZ which belongs to $-\Gamma$ and has a representation

$$(1) \qquad Y = -(Y_1 + \ldots + Y_s) \, , \qquad Y_\sigma \in C_1 \cup \ldots \cup C_k \, ,$$

with minimum s. Then $Y_1, \ldots, Y_s$ are linearly independent. Applying Theorem 3.2 to $M = -(C_1 \cup \ldots \cup C_k)$, we obtain $s \leqq n-1$. This holds also in case $Y = O$ provided that we define $s = 0$ and the void sum in (1) to mean O. It follows further that Z does not belong to the subspace E^s spanned by the points $O, Y_1, \ldots, Y_s$. For, if $Z \in E^s$, Theorem 3.2 could be applied to the cone $M' \subseteq E^s$ consisting of the rays $OY_1, \ldots, OY_s$ and with X replaced by Y. Hence, there would be a point on the segment YZ representable as a sum of less than s points of $M' \subseteq M$, in contradiction to the minimum property of s.

On the other hand, Y is relative interior to C_j since X and Z are so. Hence, Y has also a representation

$$Y = Y_1' + \ldots + Y_d' , \qquad Y_\tau' \in C_j ,$$

where $Y_1', \ldots, Y_d'$ are linearly independent and d is the dimension of C_j. Consequently, (1) yields

$$Y_1 + \ldots + Y_s + Y_1' + \ldots + Y_d' = O . \tag{2}$$

Each Y_σ, $\sigma = 1, \ldots, s$, belongs to one (or more) of the cones $C_1, \ldots, C_k$; denote this cone (or one of them) by C_{i_σ}. Applying Theorem 1.2 to the cone

$$M = C_{i_1} \cup \ldots \cup C_{i_s} \cup C_j , \tag{3}$$

we see that the lineality l of the convex hull of this cone M is greater than or equal to the maximum number of linearly independent points among the $s+d$ points occurring in (2). Hence, we have on the one hand $l \geqq d$ since $Y_1', \ldots, Y_d'$ are linearly independent, and on the other hand $l \geqq s+1$ which may be seen in the following way. Since the point Z belongs to C_j, it is a linear combination of the points $Y_1', \ldots, Y_d'$; but as shown above, it is not a linear combination of the linearly independent points $Y_1, \ldots, Y_s$. Consequently, there are at least $s+1$ linearly independent points among Y_σ, Y_τ'.

Now assume that all of the cones C_A are n-dimensional. Then we have $d = n$ and hence, $l = n$. This means that the convex hull of the cone (3) is the whole space E^n. However, because of $s \leqq n-1$ this contradicts the assumption of the theorem, and therefore all cones $C_j, j > k$, must be contained in $-\bar{\Gamma}$. This proves the theorem in this case.

If not all of the cones C_A are n-dimensional, we distinguish between two cases.

1° Suppose first that the lineality of the convex hull of any r of the cones is less than r, for every positive integer $r < n$. Then the above inequality $l \geqq s+1$ leads to a contradiction; for it expresses that the $s+1$ cones $C_{i_1}, \ldots, C_{i_s}, C_j$ have a convex hull with lineality greater than or equal to $s+1$. The conclusion is again that the cones $C_j, j > k$, are contained in $-\bar{\Gamma}$. This proves the theorem in the present case.

2° Now suppose that for some r, $1 \leqq r < n$, there are r cones, say $C_1, \ldots, C_r$, whose convex hull has lineality $l \geqq r$. Let E^r be an r-dimensional subspace of the lineality space of this convex hull, and let $E^{n-r} = E^{r*}$ be the $(n-r)$-dimensional subspace orthogonal to E^r. Project all of the cones C_A orthogonally on E^{n-r}. The projection of C_A is a closed convex cone c_A in E^{n-r}. With respect to this subspace the cones c_A satisfy the assumption of the theorem. Indeed, suppose there were $n-r$ cones c_A, say $c_{r+1}, \ldots, c_n$, whose convex hull is the entire space E^{n-r}. Then the convex hull of the cones $C_1, \ldots, C_r, C_{r+1}, \ldots, C_n$ would be the entire space E^n. The theorem being obviously valid for $n = 1$, we may proceed by induction. Thus, we assume that the statement is true in E^{n-r}, that is, there exists an $(n-r-1)$-dimensional plane E^{n-r-1} through O which divides no c_A. Then the plane E^{n-1} spanned by this E^{n-r-1} and E^r divides no C_A.

It remains to be shown that the validity of the theorem for any finite number of cones C_A implies its validity for an arbitrary infinite set $\{C_A\}$. This can be done by an argument similar to that used by F. Riesz in a proof of Helly's theorem (cf. Helly [1] and also Horn [2]).

In case all of the cones C_A are n-dimensional, define $C = \mathscr{C}(C_1 \cup \ldots \cup C_n)$ where $C_1, \ldots, C_n$ are n preassigned cones C_A. In the general case let C denote one fixed of the cones C_A. Let π be a plane which supports C, and let the ray g_π be its outer normal. All such rays make up the dual cone C^* of C. Proceeding indirectly, we assume that every plane π supporting C divides some cone C_A. Then the normal planes through O of all rays belonging to a certain neighbourhood of g_π divide the same cone C_A. By the Heine-Borel theorem, the compact set of rays C^* can be covered by finitely many of such neighbourhoods. Consequently, there are finitely many cones C_A such that every supporting plane of C divides at least one of these cones. However, this contradicts Theorem 6 for a finite number of cones.

7. The dual theorem and a theorem of Horn. Consider a set $\{C_A\}$ of closed convex cones whose dual cones $C_A{}^*$ satisfy the assumption of Theorem 6. Because of Theorem 2, this means that any n of the cones C_A have a common ray. The existence of a plane which divides no $C_A{}^*$ is equivalent to the existence of a straight line through O, namely the normal of that plane, with the property that every cone C_A contains at least one of the two rays making up this line. Observing finally that C_A has lineality 0 if $C_A{}^*$ is n-dimensional (see Section 1), we have:

THEOREM 7.1. *Let $\{C_A\}$ be a set of closed convex cones in E^n with the property that any n of the cones C_A have a common ray. Then there exists a straight line through O such that every cone C_A contains one of the two rays making up this line.*

If all of the cones C_A have lineality 0, *this line can be chosen so that one of its rays is contained in the intersection of n preassigned cones C_A.*

Except for the second statement concerning cones with lineality 0, this is equivalent to Theorem 3 of Horn's paper [2]. (Actually Horn's theorem is slightly more general since his notion of convex subset of a sphere would correspond to a notion of convex cone which comprises the complementary cones of cones convex in the usual sense.) From his Theorem 3 Horn deduces very easily, in addition to his more general Theorem 2, a generalization of Helly's theorem. In the same way this is obtained from the above Theorem 7.1 in the slightly stronger version:

THEOREM 7.2. *Let* $\{K_A\}$ *be a set of convex bodies in* E^n *with the property that any* n *of the bodies* K_A *have a common point. Further, let* $K_1, \ldots, K_n$ *be* n *arbitrary bodies* K_A*. Then through every point* P *of* E^n *there exists a straight line which intersects both the intersection* $K_1 \cap \ldots \cap K_n$ *and all of the other bodies* K_A.

REFERENCES

1. E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten,* Jber. Deutsch. Math. Verein. 32 (1923), 175–176.
2. A. Horn, *Some generalizations of Helly's theorem on convex sets,* Bull. Amer. Math. Soc. 55 (1949), 923–929.
3. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme* II, J. Reine Angew. Math. 144 (1914), 1–40.

UNIVERSITY OF LUND, SWEDEN