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ON TOTAL CURVATURES OF CLOSED SPACE CURVES

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1. Introduction.

Let C be a closed curve of class 3 in euclidean 3-space, having curvature $\varkappa(s) > 0$ and torsion $\tau(s)$, where s denotes arc-length. Define

$$\varkappa(C) = \int_C \varkappa(s)\, ds, \qquad \tau(C) = \int_C |\tau(s)|\, ds\,,$$

$$\omega(C) = \int_C \left(\varkappa^2(s) + \tau^2(s)\right)^{1/2} ds\,.$$

These quantities measure the lengths of the spherical indicatrices of the tangent vector, the binormal vector, and the principal normal vector, respectively. They are related by the inequalities

$$2\pi \leqq \varkappa(C) \leqq \omega(C) \leqq \varkappa(C) + \tau(C) \leqq 2^{1/2}\, \omega(C)\,.$$

(The first inequality was proved by Fenchel [4]. The remaining three are obvious.)

The quantity $\varkappa(C)$, known as the total curvature, has been studied by several authors (see references [3]–[7]). This paper will attempt a similar study of the quantities $\varkappa(C) + \tau(C)$ and $\omega(C)$.

Two simple closed curves belong to the same *isotopy type* if there exists an isotopy of euclidean space onto itself (not necessarily differentiable) which transforms one curve into the other. It will be proved that the greatest lower bound of $\varkappa(C) + \tau(C)$ over an isotopy class of closed curves is a number of the form $2\pi n$, n a positive integer, where $n = 1$ only for the class of unknotted curves, and where $n = 2$ only for certain especially simple knots (figure 1). On the other hand an example is given to show that the greatest lower bound of $\omega(C)$ over an isotopy class is not necessarily 2π times an integer.

If C has linking number n with some straight line, it is proved that $\varkappa(C)+\tau(C) \geqq 2\pi n$.

If the torsion $\tau(s)$ does not change sign and is not identically zero, it

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is proved that $\omega(C) \geqq 4\pi$. (This result answers a question raised by Fenchel [5, p. 51]: Does the condition $\tau(s) > 0$ imply a lower bound greater than 2π for $\varkappa(C)+\tau(C)$?)

2. The quantity $\varkappa(C)+\tau(C)$.

For each plane P through the origin let C_P denote the image of C in P under orthogonal projection. The following proposition has been proved by I. Fáry [3].

LEMMA 1. *The total curvature $\varkappa(C)$ is equal to the average over all planes P through the origin of $\varkappa(C_P)$.*

(The word average is to be interpreted as follows. To each plane P there corresponds a unique pair of unit vectors which are perpendicular to P. Therefore the usual Lebesgue measure on the unit sphere induces a measure on the set of all planes through the origin. The average may now be interpreted as a Lebesgue integral.)

A similar assertion holds for $\tau(C)$. Let $\nu(C_P)$ denote the number of points of C at which the osculating plane is perpendicular to P. This can also be interpreted as the number of points of the plane curve C_P which are either inflection points or cusps.

LEMMA 2. *$\tau(C)$ equals the average over all planes through the origin of $\pi\nu(C_P)$. Furthermore $\nu(C_P)$ is an even integer for almost all P.*

The proof of lemma 2 will depend on the following proposition of integral geometry.

LEMMA 3. *The length of a curve B on the unit sphere equals π times the average over all great circles G of the number of times that B intersects G.*

This is proved in [2], p. 81.

Now let B be the spherical indicatrix of the binormal vectors, with length $\tau(C)$. Let P be the plane spanned by G. A binormal vector lies in P if and only if the corresponding osculating plane is perpendicular to P. Therefore $\nu(C_P)$ equals the number of times that B intersects G. Since this is an even number for almost all G (all but a set of measure zero), lemma 2 follows from lemma 3.

Finally we will need a two dimensional analogue of lemma 1. Let $\mu(\boldsymbol{v}, C)$ denote the number of points of C at which the tangent vector is perpendicular to the unit vector $\boldsymbol{v}$.

LEMMA 4. *The total curvature $\varkappa(C_P)$ of a plane curve equals the average* *over the unit circle of* $\pi\mu(\boldsymbol{v}, C_P)$. *Furthermore* $\mu(\boldsymbol{v}, C_P)$ *is even for almost all* $\boldsymbol{v}$.

The proof is easily supplied. (See [7].)

For each isotopy type $\mathfrak{C}$ of closed space curves, let $(\varkappa+\tau)(\mathfrak{C})$ denote the greatest lower bound of $\varkappa(C)+\tau(C)$ over all representative curves C, which are of class 3 and satisfy $\varkappa(s) > 0$. Similarly define $\varkappa(\mathfrak{C})$ and $\omega(\mathfrak{C})$.

THEOREM 1. *For every isotopy type* $\mathfrak{C}$ *of closed curves the quantity* $(\varkappa+\tau)(\mathfrak{C})$ *is an integral multiple of* 2π.

(The corresponding assertion for $\varkappa(\mathfrak{C})$ was proved in [7]. Examples given by R. H. Fox [6] show that $\varkappa(\mathfrak{C})$ can be *any* positive integral multiple of 2π. The same examples apply to $(\varkappa+\tau)(\mathfrak{C})$.)

Suppose that $2\pi n \leqq (\varkappa+\tau)(\mathfrak{C}) < 2\pi(n+1)$. Then there exists a representative C with $\varkappa(C)+\tau(C) < 2\pi(n+1)$. We will construct isotopic curves C' with $\varkappa(C')+\tau(C')$ arbitrarily close to $2\pi n$. This will prove that $(\varkappa+\tau)(\mathfrak{C}) = 2\pi n$.

By lemmas 1 and 2 there exists a plane projection C_P of C so that $\varkappa(C_P)+\pi\nu(C_P) \leqq \varkappa(C)+\tau(C)$, where $\nu(C_P)$ is an even integer. Furthermore the plane P can be chosen so that it is not perpendicular to any tangent of C. By lemma 4 there exists a unit vector $\boldsymbol{v}$ in P so that $\pi\mu(\boldsymbol{v}, C_P) \leqq \varkappa(C_P)$, where $\mu(\boldsymbol{v}, C_P)$ is even. Since $\mu(\boldsymbol{v}, C_P) = \mu(\boldsymbol{v}, C)$ we have

$$\pi\big(\mu(\boldsymbol{v}, C)+\nu(C_P)\big) \leqq \varkappa(C_P)+\pi\nu(C_P) \leqq \varkappa(C)+\tau(C) < 2\pi(n+1)\,,$$

hence

$$\mu(\boldsymbol{v}, C)+\nu(C_P) \leqq 2n\,.$$

Choose coordinates (x_1, x_2, x_3) so that $\boldsymbol{v}$ is in the direction of the x_1-axis, and so that P is the (x_1, x_2)-plane. Define the isotopy h_t by $h_t(x_1, x_2, x_3) = (x_1, tx_2, t^2x_3)$. As $t \to 0$, the tangent vector of the curve $C_t = h_t(C)$ approaches $\pm\boldsymbol{v}$, except at the $\mu(\boldsymbol{v}, C)$ points where the tangent to C is perpendicular to $\boldsymbol{v}$. In the neighborhood of each of these points, the tangent to C_t rotates through an angle which converges to π as $t \to 0$. This proves that $\varkappa(C_t) \to \pi\mu(\boldsymbol{v}, C)$; and similarly one can prove that $\tau(C_t) \to \pi\nu(C_P)$. Consequently

$$\varkappa(C_t)+\tau(C_t) \to \pi\big(\mu(\boldsymbol{v}, C)+\nu(C_P)\big) \leqq 2\pi n\,,$$

which completes the proof.

Since $\varkappa(C) \geqq 4\pi$ for any knot (see [3], [5], or [7]), the case $(\varkappa+\tau)(\mathfrak{C}) = 2\pi$ can occur only if $\mathfrak{C}$ is the class of unknotted curves.

THEOREM 2. *If* $(\varkappa+\tau)(\mathfrak{C}) = 4\pi$, *then* $\mathfrak{C}$ *is represented by one of the curves of figure* 1.

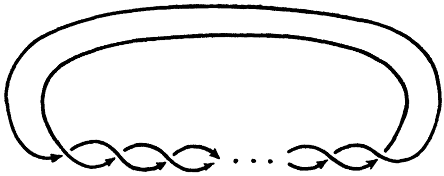

Fig. 1.

In the language of braid theory [1], these curves can be described as closed braids with two strands. The following converse to theorem 2 is evident: If $\mathfrak{C}$ can be represented by a closed braid with n strands, then $(\varkappa+\tau)(\mathfrak{C}) \leqq 2\pi n$. These assertions remain valid if knots with several components (links) are considered.

The proof follows. Choose a representative curve C and a plane projection C_P so that $\varkappa(C_P)+\pi\nu(C_P) < 6\pi$, where $\nu(C_P)$ is even. If $\nu(C_P) \geqq 2$, this would imply that $\varkappa(C_P) < 4\pi$, hence that C were unknotted. Therefore $\nu(C_P) = 0$. Since C_P is a plane curve with no inflection points, $\varkappa(C_P)$ is a multiple of 2π, and therefore $\varkappa(C_P) = 4\pi$.

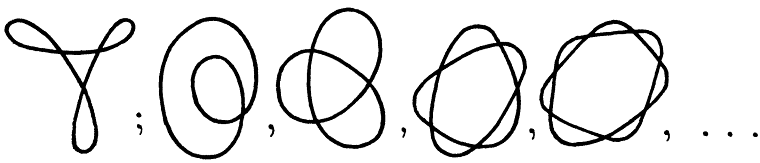

Fig. 2.

Without loss of generality we may assume that C_P has only finitely many crossing points.

A list of the possible types of plane curves having no inflection points, having total curvature 4π, and having finitely many crossing points is given in figure 2. It is not hard to show that this list is complete. Since a knot having one of these curves as plane projection is clearly isotopic to one of the curves of figure 1, this completes the proof.

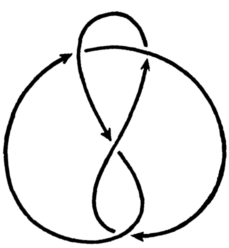

Fig. 3.

As an example to illustrate theorems 1 and 2, consider the class $\mathfrak{C}$ of the "figure eight" knot (figure 3). Since $\nu(C_P) = 2$ for the given projection C_P, and since $\mu(\boldsymbol{v}, C) = 4$ for a suitable vector $\boldsymbol{v}$ in P, it follows that $(\varkappa+\tau)(\mathfrak{C}) \leqq (2+4)\pi$, hence $(\varkappa+\tau)(\mathfrak{C}) = 2\pi, 4\pi$, or 6π. But the Alexander polynomial of $\mathfrak{C}$ is t^2-3t+1, while the curves of figure 1 have Alexander polynomials of the form $\sum_{i=0}^{2k}(-t)^i$. (For definition of the Alexander polynomial, see for example Reidemeister [8, pp. 37–41].) Therefore $\mathfrak{C}$ can not be represented by any of these curves, and we must have $(\varkappa+\tau)(\mathfrak{C}) = 6\pi$. (Note that $\varkappa(\mathfrak{C}) = 4\pi$ for this example.)

As a final application of lemmas 1 and 2, the following theorem will be proved.

THEOREM 3. *If C has linking number n with the straight line L, then $\varkappa(C)+\tau(C) \geqq 2\pi n$.*

Let P be any plane not perpendicular to L, and let L_P and C_P be the projections of L and C on P. It will be shown that $\varkappa(C_P)+\pi\nu(C_P) \geqq 2\pi n$. By lemmas 1 and 2 this will imply that $\varkappa(C)+\tau(C) \geqq 2\pi n$.

Since C_P must cross L_P at least $2n$ times, it follows that the tangent to C_P must be parallel to L_P at least $2n$ times. Consider two consecutive points of C_P for which the tangents are parallel. If the tangents point in opposite directions, then the total curvature of the curve segment between the two points is at least π. If the tangents point in the same direction, then there must be some inflection point in between. In either case there is a contribution of at least π to $\varkappa(C_P)+\pi\nu(C_P)$. Therefore $\varkappa(C_P)+\pi\nu(C_P) \geqq 2\pi n$, which completes the proof.

3. The quantity $\omega(C)$.

THEOREM 4. *If $\tau(s) \geqq 0$ for all s, but $\tau(s)$ is not identically 0, then $\omega(C) \geqq 4\pi$.*

An example given by Fenchel [5] shows that $\omega(C)$ can be arbitrarily close to 4π.

It is first necessary to prove a lemma and to review some formulas of differential geometry. Suppose that two antipodal points of the unit sphere have been specified as north and south poles.

LEMMA 5. *Let G and G' be oriented great circles, such that G does not pass through the north and south poles. If the tangent vector to G', at the point where G' crosses south of G, lies north of the equator, then G' makes a smaller angle with the equator than G.*

Since the crossing point lies on the segment between the southernmost point of G' and the northernmost point, it follows that the southernmost point of G' lies north of G. Hence G' makes a smaller angle with the equator than G.

Let T denote the spherical indicatrix of the tangent vectors of C. The tangent, principal normal, and binormal vectors $\boldsymbol{t}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ are related by the Frenet formulas:

$$\begin{aligned} d\boldsymbol{t}/ds &= \qquad\ \varkappa\boldsymbol{n} \\ d\boldsymbol{n}/ds &= -\varkappa\boldsymbol{t} \qquad +\tau\boldsymbol{b} \\ d\boldsymbol{b}/ds &= \qquad -\tau\boldsymbol{n}\ . \end{aligned} \tag{1}$$

In particular the tangent vector of the curve T is just $\boldsymbol{n}$. Considered as a spherical curve, T has geodesic curvature $\tau/\varkappa$, where a positive value indicates that T curves towards the binormal vector $\boldsymbol{b}$.

The proof of theorem 4 follows. By lemma 3 it is sufficient to prove that the principal normal vector $\boldsymbol{n}(s)$ intersects every great circle at least four times. Suppose that it intersected some great circle E only twice, at $\boldsymbol{n}(s_0)$ and $\boldsymbol{n}(s_1)$. Taking E as equator, let $-\pi/2 \leqq \varphi(\boldsymbol{v}) \leqq \pi/2$ denote the latitude (angle north of equator) of any point $\boldsymbol{v}$. By proper choice of s_0, s_1, and the direction north, we may assume that

$$|\varphi(\boldsymbol{b}(s_0))| \geqq |\varphi(\boldsymbol{b}(s_1))| \tag{2}$$

and

$$\varphi(\boldsymbol{n}(s)) > 0 \qquad \text{for } s_0 < s < s_1\ . \tag{3}$$

Let G be the great circle which is tangent to T at $\boldsymbol{t}(s_0)$. It will be shown that T cannot cross G; hence that T cannot be the tangent indicatrix of a closed curve (see for example [5, p. 49]). By symmetry it will be sufficient to restrict our attention to the interval $s_0 \leqq s \leqq s_1$.

Formulas (1) and (3) imply that $\varphi(\boldsymbol{b}(s))$ is a monotone decreasing function of s; strictly decreasing whenever $\tau(s) > 0$. Together with (2) this implies

$$|\varphi(\boldsymbol{b}(s_0))| \geqq |\varphi(\boldsymbol{b}(s))|\ . \tag{4}$$

If $\varphi(\boldsymbol{b}(s_0))$ were equal to $\varphi(\boldsymbol{b}(s_1))$, it would follow that $\tau(s)$ were identically zero, not only for $s_0 \leqq s \leqq s_1$, but also, by a similar argument, for the remaining values of s. Since this is excluded, it follows that $\varphi(\boldsymbol{b}(s_0)) > \varphi(\boldsymbol{b}(s_1))$; hence by (2):

$$\varphi(\boldsymbol{b}(s_0)) > 0\ . \tag{5}$$

Suppose that T crosses G at some point $\boldsymbol{t}(s')$ with $s_0 < s' < s_1$. Let $\boldsymbol{t}(s')$ be the first such crossing, and let G' be the great circle which is tangent to T at $\boldsymbol{t}(s')$. It will be proved that G and G' satisfy the hypothesis of lemma 5. First note that T cannot be tangent to G at a crossing: this follows from the fact that the geodesic curvature $\tau/\varkappa$ of T cannot change sign. Therefore the great circles G and G' are distinct. According to (5), $\boldsymbol{b}(s_0)$ lies north of the equator. Since $\boldsymbol{b}(s_0)$ is the pole of the great circle G, this implies that G does not pass through the north and south poles. Since the geodesic curvature of T is non-negative, T must always either curve towards $\boldsymbol{b}(s)$ or else follow a great circle. Therefore T either curves north of G at $\boldsymbol{t}(s_0)$, or else follows the circle G for some interval and then curves north. In either case T, and therefore G', must cross south of G at $\boldsymbol{t}(s')$. The tangent to G' at $\boldsymbol{t}(s')$ is just $\boldsymbol{n}(s')$, which lies north of the equator by (3).

Now by lemma 5, G' makes a smaller angle with the equator than G. But the poles of the great circles G' and G are just the points $\boldsymbol{b}(s')$ and $\boldsymbol{b}(s_0)$. Therefore

$$\left|\varphi\big(\boldsymbol{b}(s')\big)\right| > \left|\varphi\big(\boldsymbol{b}(s_0)\big)\right| ,$$

which contradicts (4), and completes the proof.

In conclusion an example will be given to show that $\omega(\mathfrak{C})$ is not necessarily an integral multiple of 2π. In fact for the class $\mathfrak{C}$ of the figure eight knot (figure 3) it will be shown that

$$4\pi < \omega(\mathfrak{C}) < 6\pi .$$

For any representative curve C, the inequality $\varkappa(C)+\tau(C) \geqq 6\pi$ implies that

$$\omega(C) \geqq \big(\varkappa(C)+\tau(C)\big)2^{-1/2} \geqq 6\pi\, 2^{-1/2} > 4\pi .$$

(Actually the slightly better result $\omega(C) > 2\pi\, 5^{1/2}$ may be obtained by also using the inequalities $\omega(C) \geqq \big(\varkappa^2(C)+\tau^2(C)\big)^{1/2}$ and $\varkappa(C) > 4\pi$.) It is now necessary to construct a representative curve C having $\omega(C) < 6\pi$.

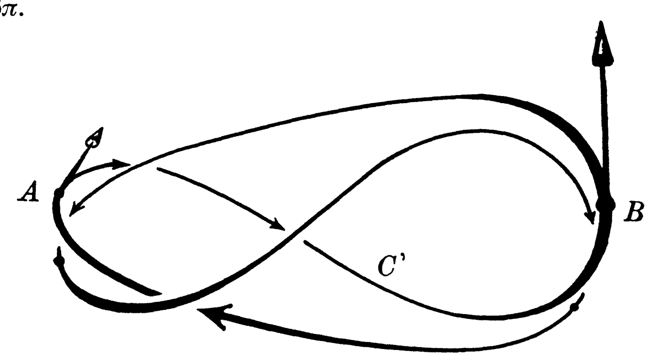

Fig. 4.

Consider curves of the form illustrated in figure 4, composed of the arc segment C' joining points A and B, and of three congruent arc segments (one properly congruent, and two congruent with reversed orientations). The tangent vectors $\boldsymbol{t}_A$, $\boldsymbol{t}_B$, and the displacement vector $\overrightarrow{AB}$ are to be perpendicular to each other, up to an ε approximation. To compute $\omega(C')$, it is convenient to go over to the spherical indicatrix T of the tangent vectors. Since the curve T has tangent vector $\boldsymbol{n}(s)$, it follows that $\omega(C) = \varkappa(T)$ and $\omega(C') = \varkappa(T')$. Thus it is necessary to find a curve segment T' joining $\boldsymbol{t}_A$ and $\boldsymbol{t}_B$ on the unit sphere, having total curvature as small as possible, subject to the condition $\int_{T'} \boldsymbol{t}(s)\, ds = \overrightarrow{AB}$.

This last condition may also be expressed as follows. The vector $\overrightarrow{AB}$ should be an interior point of the convex cone generated by the curve segment T'. Take T' to be a circle arc with $\boldsymbol{t}_A$ and $\boldsymbol{t}_B$ as end points, and $\overrightarrow{AB}$ as interior point. Letting the mid-point of this circle arc approach $\overrightarrow{AB}$, the total curvature $\varkappa(T')$ can be brought arbitrarily close to $\frac{2}{3} \cdot 2\pi$. Now piecing the arc T' together with three congruent arcs (two of which have reversed orientations) we obtain a closed spherical curve T_1 of class 1, with $\varkappa(T_1)$ arbitrarily close to $4 \cdot \frac{2}{3} \cdot 2\pi$. This curve T_1 can clearly be approximated by a curve T of class 2. If T is described by the vector $\boldsymbol{t}(u)$, then, for a suitably chosen weight function $w(u) > 0$, the vector $\boldsymbol{x}(u) = \int_0^u w(u)\, \boldsymbol{t}(u)\, du$ will describe a closed space curve C, of class 3 with $\varkappa(s) > 0$, belonging to the isotopy class $\mathfrak{C}$, and having $\omega(C) = \varkappa(T)$ arbitrarily close to $\frac{8}{3} \cdot 2\pi$. Thus it has been proved that

$$4\pi < 2 \cdot 5^{1/2}\pi \leqq \omega(\mathfrak{C}) \leqq \tfrac{8}{3} \cdot 2\pi < 6\pi \,.$$

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For further references see [5].

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