A RECIPROCITY FORMULA
FOR WEIGHTED QUADRATIC PARTITIONS

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1. Introduction. Let \( q = p^n \), where \( p \) is an odd prime. For \( \alpha \in GF(q) \), put
\[
e(\alpha) = e^{2\pi i \frac{t(\alpha)}{p}},
\]
where \( t(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^n-1} \);
also let
\[
S(\alpha, \lambda, Q) = \sum_{Q(\xi) = \alpha} e(2\lambda_1 \xi_1 + \cdots + 2\lambda_r \xi_r),
\]
where \( \alpha, \lambda_j \in GF(q) \),
\[
Q(u) = \sum_{1}^{r} \alpha_{kj} u_k u_j \quad (\alpha_{kj} \in GF(q), \delta = |\alpha_{kj}| \neq 0),
\]
and the summation in the right member of (1.2) is over all \( \xi_i \in GF(q) \) such that \( Q(\xi_1, \ldots, \xi_r) = \alpha \). It was shown incidentally in [2] that if
\[
\lambda_k = \sum_{j=1}^{r} \alpha_{kj} \lambda_j' \quad (k = 1, \ldots, r)
\]
then \( S(\alpha, \lambda, Q) \) satisfies the following reciprocity relation,
\[
S(\alpha, \lambda, Q) = S(\alpha, \lambda', Q'),
\]
where \( Q'(u) \) denotes the quadratic form inverse to \( Q(u) \). In this note we give a direct proof of (1.4) as well as of one or two extensions. We also consider the analogous formula when the coefficients are rational integers.

2. By a well-known theorem [1, p. 160, Theorem 3] the linear transformation
\[
\xi_k = \sum_{j=1}^{r} \alpha_{kj} \xi_j'
\]
carries \( Q \) into \( Q' \), that is
\[
Q(\xi') = Q'(\xi).
\]
We have also
\[
\sum_{j=1}^{r} \lambda_j \xi_j' = \sum_{j=1}^{r} \lambda_j' \xi_j .
\]

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Now by (1.2),
\[ S(x, \lambda, Q) = \sum_{Q(\xi) = x} e(2\lambda_1\xi_1' + \ldots + 2\lambda_r\xi_r'), \]
and by (2.2) and (2.3) this becomes
\[ S(x, \lambda, Q) = \sum_{Q(\xi) = x} e(2\lambda_1'\xi_1 + \ldots + 2\lambda_r'\xi_r) = S(x, \lambda', Q'), \]
which evidently proves (1.4).

If \( f(u) = f(u_1, \ldots, u_r) \) denotes an arbitrary polynomial with coefficients in \( GF(q) \), we define
\[ S(x, f, Q) = \sum_{Q(\xi) = x} e(f(\xi)), \]
which clearly generalizes (1.2). Now let (2.1) carry \( f \) into \( f' \), that is,
\[ f(\xi') = f'(\xi), \]
thus generalizing (2.3). Then it is clear that the previous argument may be applied to yield the formula
\[ S(x, f, Q) = S(x, f', Q'). \]
We have thus obtained a first generalization of (1.4). However this can be carried a bit further. Let \( g(u) = g(u_1, \ldots, u_r) \) denote another arbitrary polynomial with coefficients in \( GF(q) \) and let (2.1) carry \( g \) into \( g' \), that is,
\[ g(\xi') = g'(\xi). \]
We define
\[ S(x, f, g) = \sum_{g(\xi) = x} e(f(\xi)), \]
the summation extending over all \( \xi_i \in GF(q) \) such that \( g(\xi_1, \ldots, \xi_r) = x \). Then exactly as in the proof of (1.4) we have
\[ S(x, f, g) = \sum_{g(\xi) = x} e(f(\xi')) = \sum_{g'(\xi) = x} e(f'(\xi')), \]
which implies
\[ S(x, f, g) = S(x, f', g'). \]
Thus (2.9) together with (2.5) and (2.7) furnish a two-fold generalization of (1.4). Note that it is no longer necessary to assume \( p \) odd.

3. We now briefly consider an analog of (1.4) involving positive quadratic forms with rational integral coefficients. Let
\[ Q(u) = \sum_{1}^{r} a_{kj}u_ku_j \quad (|a_{kj}| = 1), \]
where the $a_{kj}$ are rational integers. We assume that $Q(u)$ is a positive definite form; thus the equation $Q(u) = m$, where $m$ is an arbitrary positive integer, has at most a finite number of integral solutions. We define

$$S(m, \lambda, Q) = \sum_{Q(u) = m}^\prime \exp\left(2\pi i(\lambda_1 u_1 + \ldots + \lambda_r u_r)\right),$$

where the $\lambda_j$ denote arbitrary complex numbers. If we put

$$u_k = \sum_j a_{kj}^\prime u_j^\prime,$$

then in view of the hypothesis $|a_{kj}| = 1$, the inverse of (3.3) also has integral coefficients; also, as in (2.2), we have now

$$Q(u') = Q'(u),$$

where again $Q'$ denotes the quadratic form inverse to $Q$. If we define $\lambda_k'$ by means of

$$\lambda_k = \sum_{1}^r a_{kj}^\prime \lambda_j^\prime,$$

then exactly as in § 2 we may prove the reciprocity formula

$$S(m, \lambda, Q) = S(m, \lambda', Q').$$

Clearly (3.6) can be generalized but we shall not take the space to do so.

The following remark may be of interest. Define

$$\vartheta(t, \lambda, Q) = \sum_{m=0}^\infty S(m, \lambda, Q)e^{-mt}$$

$$= \sum_{u_1, \ldots, u_r = -\infty}^\infty \exp\left(-tQ(u) + 2\pi i(\lambda_1 u_1 + \ldots + \lambda_r u_r)\right),$$

where $\text{Re}(t) > 0$. Applying (3.6), we see that (3.7) yields the formula

$$\vartheta(t, \lambda, Q) = \vartheta(t, \lambda', Q'),$$

subject to (3.4), (3.5) and the stated hypothesis for $Q$.

REFERENCES

2. L. Carlitz, Weighted quadratic partitions over a finite field, Canadian J. Math. 5 (1953), 317–323.

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