TWO SUMMATION FORMULAE
FOR PRODUCT SUMS OF BINOMIAL COEFFICIENTS

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It is well known that the formula

\[ \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n} \]

holds for arbitrary \( a \) and \( b \) and integral \( n \). A simple way of proving it, is to consider the relation

\[ (1+x)^a(1+x)^b = (1+x)^{a+b}. \]

If we introduce the binomial series for \( (1+x)^a, (1+x)^b \), and \( (1+x)^{a+b} \) and compare the coefficients of \( x^n \) on both sides of (2), then (1) follows immediately.

Since I have found no formula for the sum

\[ \sum_{i=0}^{k} \binom{a}{i} \binom{b}{n-i}, \quad k = 1, \ldots, n-1, \]

in the literature, I think that the following two formulæ may be of some interest. They cover the special cases \( b = -a \) and \( b = 1-a \) and were developed for use in a problem in probability. (See the following, paper, pp. 276 and 278.)

The formulæ, will be proved by induction, are:

\[ \sum_{i=0}^{k} \binom{a}{i} \binom{-a}{n-i} = \frac{n-k}{n} \binom{a-1}{k} \binom{-a}{n-k}, \quad n = 1, 2, \ldots, \]

\( k = 0, 1, \ldots, n, \)

and

\[ \sum_{i=0}^{k} \binom{a}{i} \binom{1-a}{n-i} = \frac{(n-1)(1-a) - k(a-1)}{n(n-1)} \binom{a-1}{k} \binom{-a}{n-k-1}, \]

\( n = 2, 3, \ldots, \quad k = 0, 1, \ldots, n-1. \)

Proof of (5): For \( n = 1, 2, \ldots \) and \( k = 0 \) the formula evidently holds, since both sides equal \( \binom{-a}{n} \). We assume that (5) holds for \( n (\geq 1) \) and \( k (\leq n) \) and get

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\[
\sum_{i=0}^{k+1} \binom{n-i}{i} \binom{-a}{n-i} = \frac{n-k}{n} \binom{a-1}{k} \binom{-a}{n-k} + \binom{a}{k+1} \binom{-a}{n-k-1}
\]

\[
= \binom{a-1}{k} \binom{-a}{n-k-1} \left( \frac{(n-k)(-a-n+k+1)}{n(n-k)} + \frac{a}{k+1} \right)
\]

\[
= \binom{a-1}{k} \binom{-a}{n-k-1} \frac{(a-k-1)(n-k-1)}{n(k+1)} = \frac{n-k-1}{n} \binom{a-1}{k+1} \binom{-a}{n-k-1},
\]

and (5) follows by induction with respect to \(k\).

Proof of (6): For \(n = 2, 3, \ldots\) and \(k = 0\) formula (6) holds since both sides equal \(\binom{1-a}{n}\). We assume that (6) holds for \(n (\geq 2)\) and \(k (< n-1)\) and get

\[
\sum_{i=0}^{k+1} \binom{n-i}{i} \binom{1-a}{n-i}
\]

\[
= \frac{(n-1)(1-a)-k}{n(n-1)} \binom{a-1}{k} \binom{-a}{n-k-1} + \binom{a}{k+1} \binom{1-a}{n-k-1}
\]

\[
= \binom{a-1}{k} \binom{-a}{n-k-2} \left\{ \left[ \frac{(n-1)(1-a)-k}{n(n-1)(n-k-1)} \right] \left[ (-a-n+k+2) + \frac{a(1-a)}{(k+1)(n-k-1)} \right] \right\}
\]

\[
= \binom{a-1}{k} \binom{-a}{n-k-2} \frac{(n-k-1)[(n-1)(1-a)-k-1](a-k-1)}{n(n-1)(k+1)(n-k-1)}
\]

\[
= \frac{(n-1)(1-a)-k-1}{n(n-1)} \binom{a-1}{k+1} \binom{-a}{n-k-2}.
\]

By induction with respect to \(k\) the formula (6) follows for \(k = 1, 2, \ldots, n-1\). For \(k = n-1\) the operations in (8) are invalid.

It will be noted in both proofs that for each \(n\) the induction only proceeds through a finite number of steps.