CONGRUOUS AND
INCONGRUOUS ONE-TO-ONE CORRESPONDENCES

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In this note we first prove a general theorem concerning one-to-one correspondences between a set and itself, relative to decompositions of the set into subsets, and then obtain some related results dealing with more special correspondences involving real or rational numbers.

Let \( I \) be a one-to-one correspondence defined between the elements of the (not necessarily distinct) sets \( M \) and \( N \), or a one-to-one transformation defined on \( M \) and mapping \( M \) onto \( N \), and let \( m \in M \) and \( n \in N \) be (not necessarily distinct) corresponding elements under \( I \). If \( m \neq n \), we call \( m \) a free element of \( M \), under \( I \). There are obviously the same number of free elements, under \( I \), in \( M \) as in \( N \). We say that \( I \) has \( F \) free elements, if the cardinal number of the set of free elements in \( M \), under \( I \), is \( F \). Let \( \mathcal{F} \) be an arbitrary class of mutually exclusive sets, with \( |\mathcal{F}| \geq 2 \), and let \( p \geq 1 \). If \( m \) and \( n \) belong to the same element of \( \mathcal{F} \), we call them a congruous pair of elements relative to \( \mathcal{F} \); if they belong to distinct element of \( \mathcal{F} \), we call them an incongruous pair of elements relative to \( \mathcal{F} \). We say that \( I \) is

(a) congruous,
(b) \( p \)-congruous,
(c) at least \( p \)-incongruous,
(d) \( p \)-incongruous,

relative to \( \mathcal{F} \), according as the following conditions are satisfied:

(a) all pairs of corresponding elements are congruous relative to \( \mathcal{F} \);
(b) \( I \) is congruous relative to \( \mathcal{F} \), and every element of \( \mathcal{F} \) contains exactly \( p \) pairs of corresponding elements;
(c) if \( X \in \mathcal{F} \), \( Y \in \mathcal{F} \), \( X \neq Y \), then there are at least \( p \) incongruous pairs of corresponding elements relative to the class \( \{ X, Y \} \);
(d) if \( X \in \mathcal{F} \), \( Y \in \mathcal{F} \), \( X \neq Y \), then there are exactly \( p \) incongruous pairs of corresponding elements relative to the class \( \{ X, Y \} \).

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A $\mathfrak{p}$-decomposition ($\mathfrak{p} \geq 1$) of a set $E$ is a class, $\mathfrak{B}$, of $\mathfrak{p}$ nonempty mutually exclusive sets whose union is $E$.

$C$ denotes an ordered set whose order type is that of the real numbers in their natural order. $R$ denotes an ordered set whose order type is that of the rational numbers in their natural order, except in Theorem 5, where $R$ stands for the set of rational points on a straight line.

By an interval of an ordered set $S$ we mean either $S$ itself, or the subset of $S$ preceding some element of $S$, or the subset of $S$ succeeding some element of $S$, or the subset of $S$ between two distinct elements of $S$. A subset, $S'$, of $S$ is dense in $S$ provided that every interval of $S$ contains an element of $S'$.

**Theorem 1.** Let $\mathcal{G}$ be a class of $\aleph_\alpha$ one-to-one correspondences each of which has at least $\aleph_\alpha$ free elements, $E$ be the union of the sets between which the elements of $\mathcal{G}$ are defined, and $\mathfrak{p}$ be a cardinal number with $2 \leq \mathfrak{p} \leq \aleph_\alpha$. Then there exists a $\mathfrak{p}$-decomposition, $\mathfrak{B}$, of $E$ such that every element of $\mathcal{G}$ is at least $\aleph_\alpha$-incongruous relative to $\mathfrak{B}$.

**Proof:** Consider $\aleph_\alpha$ replicas of every element of $\mathcal{G}$, well-order the resulting complex of $\aleph_\alpha^2 = \aleph_\alpha$ one-to-one correspondences to form a sequence

$$G_0, G_1, \ldots, G_\xi, \ldots \quad (\xi < \omega_\alpha),$$

and denote by $M_\xi, N_\xi$ the (not necessarily distinct) sets between which $G_\xi (\xi < \omega_\alpha)$ is defined. Let $\varrho$ be the smallest ordinal number such that $|\varrho| = \mathfrak{p}$. The power of the set of all ordered pairs, $(\gamma, \delta)$, of ordinal numbers with $\gamma < \delta < \varrho$, is not greater than $\aleph_\alpha$. Consider $\aleph_\alpha$ replicas of each of these ordered pairs, and well-order the resulting complex of $\aleph_\alpha$ ordered pairs to form a sequence

$$p_0, p_1, \ldots, p_\xi, \ldots \quad (\xi < \omega_\alpha).$$

Let $m_0 \in M_0, n_0 \in N_0$ be free corresponding elements under $G_0$. Suppose that $0 < \xi < \omega_\alpha$, and that elements $m_\sigma \in M_\sigma, n_\sigma \in N_\sigma$ have been defined for every $\sigma < \xi$. If we put $V_\xi = \{m_\sigma\}_{\sigma < \xi} \cup \{n_\sigma\}_{\sigma < \xi}$, then $|V_\xi| < \aleph_\alpha$. Let $m_\xi \in M_\xi - V_\xi, n_\xi \in N_\xi - V_\xi$ be free corresponding elements under $G_\xi$; they exist because $G_\xi$, by hypothesis, has at least $\aleph_\alpha$ free elements. The set $V_{\omega_\alpha} = \{m_\xi\}_{\xi < \omega_\alpha} \cup \{n_\xi\}_{\xi < \omega_\alpha}$ is thus defined by transfinite induction.

We express $V_{\omega_\alpha}$ as the union of $\mathfrak{p}$ mutually exclusive sets $B_\pi (\pi < \varrho)$, as follows: Suppose that $\xi < \omega_\alpha$. The terms of (1) which are identical with $G_\xi$ but whose subscripts are less than $\xi$, form a subsequence, of some order type $\tau_\xi < \omega_\alpha$, of (1). If $p_\tau_\xi = (\gamma, \delta)$, assign $m_\xi$ to $B_\gamma$ and $n_\xi$ to $B_\delta$.

Math. Seand. 1. 17
Express $E - (\{m_\xi\}_{\xi < \omega_\alpha} \cup \{n_\xi\}_{\xi < \omega_\alpha})$ in an arbitrary manner as the union of $\varphi$ mutually exclusive sets $B_{\pi'} (\pi < \varphi)$, and put $A_\pi = B_\pi \cup B_{\pi'} (\pi < \varphi)$. The class $\mathfrak{P} = \{A_\pi\}_{\pi < \varphi}$ is a $\varphi$-decomposition of $E$.

Now let $\Gamma \in \mathfrak{G}$, and suppose that $A_\gamma$ and $A_\delta (\gamma < \delta < \varphi)$ are any two distinct elements of $\mathfrak{P}$. This $\Gamma$ appears $\kappa_\alpha$ times in (1); let

$$\Gamma_{\mu_\xi}, \Gamma_{\nu_\xi}, \ldots, \Gamma_{\mu_\xi'}, \ldots \quad (\xi < \omega_\alpha)$$

be the subsequence of (1), whose terms are identical with $\Gamma$. The ordered pair $(\gamma, \delta)$ appears $\kappa_\alpha$ times in (2); let

$$p_{\mu_\xi}, p_{\nu_\xi}, \ldots, p_{\mu_\xi'}, \ldots \quad (\xi < \omega_\alpha)$$

be the subsequence of (2), whose terms are identical with $(\gamma, \delta)$. According to the definition of the sets $B_\pi (\pi < \varphi)$, we have $m_{\mu_\xi} \in B_\gamma$, $n_{\nu_\xi} \in B_\delta$ ($\xi < \omega_\alpha$), which means that $\Gamma$ is at least $\kappa_\alpha$-incongruous relative to the class $\{A_\gamma, A_\delta\}$. This completes the proof of Theorem 1.

**Theorem 2.** Let $2 \leq \varphi \leq 2^{\kappa_\alpha}$. Then there exists a $\varphi$-decomposition, $\mathfrak{P}$, of $C$, such that every antisimilarity or nonidentical similarity between two (not necessarily distinct) intervals of $C$ is $2^{\kappa_\alpha}$-incongruous relative to $\mathfrak{P}$.

**Proof:** Let $\mathfrak{G}$ be the class of all antisimilarities and nonidentical similarities between all pairs of (not necessarily distinct) intervals of $C$. There are $2^{\kappa_\alpha}$ intervals of $C$, and $2^{\kappa_\alpha}$ antisimilarities and nonidentical similarities defined between every pair of these intervals. It follows that $|\mathfrak{G}| = 2^{\kappa_\alpha}$. Every element of $\mathfrak{G}$, since it is not the identity, has a whole interval of free elements. If we now put $\kappa_\alpha = 2^{\kappa_\alpha}$ and $E = C$, then the hypotheses of Theorem 1 are satisfied, and Theorem 2 is an immediate consequence of the conclusion of Theorem 1.

Let $R'$ be a set which is similar to $R$, $R$ be the union of the mutually exclusive sets $A_\nu (\nu < \varphi, 1 \leq \varphi \leq \omega)$ each of which is dense in $R$, and $R'$ be the union of the mutually exclusive sets $A_\nu' (\nu < \varphi)$ each of which is dense in $R'$. According to Skolem [1, pp. 30–36], there exists a similarity mapping, $\Gamma$, of $R$ onto $R'$, such that $\Gamma(A_\nu) = A_\nu'$ for every $\nu < \varphi$. Let $B_\varphi$ be a subset of $A_\varphi$ which is cofinal with $A_\varphi$ and has order type $\omega$. One can obtain $2^{\kappa_\alpha}$ subsets of $A_\varphi'$ each of which is cofinal with $A_\varphi'$ and has order type $\omega$. Let $B_\varphi'$ denote one of these subsets of $A_\varphi'$. Then it is not difficult to see that the $\Gamma$ in Skolem's theorem can be chosen so as to have the additional property that $\Gamma(B_\varphi) = B_\varphi'$. Furthermore, each choice of $B_\varphi'$ leads to a different $\Gamma$. Thus, as an immediate consequence of Skolem's theorem, we have
Theorem 3. Let $\mathfrak{B}$ be a $p$-decomposition of $R$, where $1 \leq p \leq \aleph_0$, such that every element of $\mathfrak{B}$ is dense in $R$. Then, between any two (not necessarily distinct) intervals of $R$, there exist $2^{\aleph_0}$ similarities which are $\aleph_0$-congruous relative to $\mathfrak{B}$.

Theorem 3 may no longer hold if we drop the condition that every element of $\mathfrak{B}$ be dense in $R$. To see this, it suffices to well-order the elements of $R$ to form a sequence, $r_0, r_1, \ldots, r_v, \ldots$ ($v < \omega$), put $A_v = \{r_v\}$ for every $v < \omega$, and take $\mathfrak{B} = \{A_v\}_{v < \omega}$. Then every similarity (except the identity) defined on any interval of $R$ is 1-incongruous relative to some infinite subclass (depending on the similarity) of $\mathfrak{B}$. If, however, $p$ is finite, then it is possible to obtain the following result:

Theorem 4. Let $\mathfrak{B}$ be a $p$-decomposition of $R$, where $1 \leq p < \aleph_0$. Then, between every interval of $R$ and itself, there exist $2^{\aleph_0}$ similarities which are congruous relative to $\mathfrak{B}$.

Proof: Let $I$ be an arbitrary interval of $R$. Since $R$ is the union of the finitely many elements of $\mathfrak{B}$, there exists an interval, $I_1$, of $I$ such that, for some element, say $A_1$, of $\mathfrak{B}$, $A_1 \cap I_1$ is dense in $I_1$. Let $A_2, A_3, \ldots, A_q$, where $|q| = p$, be the remaining elements of $\mathfrak{B}$, in case $p > 1$. We define an interval $I_q$, by induction, as follows: Suppose that $1 \leq v < q$ and that the intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_v$ have been defined so that, if $1 \leq r \leq v$, $A_r \cap I_v$ is either dense in $I_v$ or empty. If there exists an interval, $J$, of $I_v$ such that $A_{v+1} \cap J = 0$, we define $I_{v+1}$ to be $J$; otherwise, we put $I_{v+1} = I_v$, in which case $A_{v+1} \cap I_{v+1}$ is dense in $I_{v+1}$. Let $B_v = A_v \cap I_q (1 \leq v \leq q)$; evidently $B_v$ is either dense in $I_q$ or empty, and $B_1$, certainly, is dense in $I_q$. According to Theorem 3, there exist $2^{\aleph_0}$ similarities between $I_q$ and itself, which are congruous relative to that subclass of $\mathfrak{B}$ consisting of those elements $A_v$ for which the corresponding set $B_v$ is not empty. Each of these similarities when extended to $I$ so as to be the identity on $I - I_q$, is congruous relative to $\mathfrak{B}$, and hence the proof of the theorem is complete.

Let $R$ be the set of rational points on a straight line. There are $\aleph_0$ displacements between $R$ and itself, and at most 2 displacements between two (not necessarily distinct) congruent intervals of $R$ if these intervals are different from $R$. If such a displacement is not the identity, it has $\aleph_0$ free elements. A simple application of Theorem 1 yields

Theorem 5. If $2 \leq p \leq \aleph_0$, then there exists a $p$-decomposition, $\mathfrak{B}$, of $R$, such that every nonidentical displacement between two (not necessarily distinct) congruent intervals of $R$ is $\aleph_0$-incongruous relative to $\mathfrak{B}$.
REFERENCE


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