ON THE ASYMPOTOTIC DISTRIBUTION
OF THE EIGENVALUES AND EIGENFUNCTIONS
OF ELLIPTIC DIFFERENTIAL OPERATORS

LARS GÅRDING

Introduction. Let \( a = a(x, D) \) be a differential operator of the form
\[
a(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha \quad (m \geq 1),
\]
where \( x = (x_1, \ldots, x_n) \) is a point in real \( n \)-space, \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a
differentiation index, \( |\alpha| = \alpha_1 + \ldots + \alpha_n \), and
\[
D^\alpha = D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]
The coefficients \( a_\alpha(x) \) are supposed to be infinitely differentiable in an
open region \( T \), and the operator \( a \) is supposed to be elliptic so that
\[
a_0(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \quad (\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n})
\]
is a positive definite polynomial in \( \xi \) for all \( x \) in \( T \).

Let \( H = H(S) \) be the set of all infinitely differentiable functions vanishing
outside compact subsets of an open bounded set \( S \) whose closure is
contained\(^{1}\) in \( T \). Put
\[
(f, g) = \int_S \sum_{|\alpha| \leq m} f_\alpha(x) \overline{g_\alpha(x)} \, dx, \quad ||f||^2 = ((f, f)), \quad f_\alpha = D^\alpha f,
\]
and
\[
(f, g) = \int_S f(x) \overline{g(x)} \, dx, \quad |f|^2 = (f, f).
\]
Closing \( H(S) \) in the norm \( ||f|| \) we get a Hilbert space \( \mathcal{H} = \mathcal{H}(S) \) which may
be described roughly as the set of all functions in \( S \) having square integrable
derivatives of any order \( \leq m \), those of order \( < m \) vanishing at
the boundary of \( S \).

Received August 3, 1953.

\(^{1}\) The assumptions on the differential operator \( a \) may be weakened by considering it
only in \( S \) and by letting the coefficients be only sufficiently differentiable, but we do not
go into the details.
Consider \((af, g)\) where \(f\) and \(g\) are in \(H(S)\). By integrations by parts it may be written in the form
\[
a(f, g) = \sum_S \alpha' \alpha(x) f_\beta(x) g_\beta(x) \, dx \quad (|x| \leq m, |\beta| \leq m),
\]
and hence it is obviously a bounded function of \(f\) and \(g\) in \(\mathcal{H}\). As shown by the author [3], the form \(a(f, g)\) is also bounded from below in the following sense. Let \(t\) be a large positive number and put
\[
a_t(f, g) = a(f, g) + t(f, g)
\]
and
\[
((f, g))_t = ((f, g)) + t(f, g).
\]
Then there exist a number \(t_0\) and a number \(c > 0\) so that\(^2\)
\[
(0.3) \quad c^{-1} \langle (f, f) \rangle_t \leq |a_t(f, f)| \leq c \langle (f, f) \rangle_t \quad (t > t_0)
\]
for all \(f\) in \(\mathcal{H}\). Hence the bounded linear operator \(N_t\) from \(\mathcal{H}\) to \(\mathcal{H}\) defined by
\[
a_t(f, f') = \langle (N_t f, f') \rangle_t, \quad f, f', N_t f \in \mathcal{H},
\]
has a bounded inverse \(N_t^{-1}\). Let \(\mathcal{H}_0 = \mathcal{H}_0(S)\) be the set of all square integrable functions defined in \(S\). As is easily seen, the equation
\[
(f, f') = \langle (M_t f, f') \rangle_t, \quad f \in \mathcal{H}_0, \quad M_t f, f' \in \mathcal{H},
\]
defines a completely continuous linear operator from \(\mathcal{H}_0\) to \(\mathcal{H}\). Hence
\[
(f, f') = a_t(G_t f, f'), \quad f \in \mathcal{H}_0, \quad f' \in \mathcal{H},
\]
where \(G_t = N_t^{-1} M_t\), defines a completely continuous linear operator \(G_t\) from \(\mathcal{H}_0\) to \(\mathcal{H}\) which will be called Green's transformation corresponding to the differential operator \(a_t = a + t\) and the linear subset \(\mathcal{H}\) of \(\mathcal{H}_0\). The reason is that \(G_t\) transforms \(\mathcal{H}_0\) into \(\mathcal{H}\) and that \(G_t^{-1}\) is an extension of the differential operator \(a_t\) whose graph is the set of all pairs \(\{f, a_tf\}\) where \(f \in \mathcal{H}\) is \(2m\) times continuously differentiable and \(a_t f\) is in \(\mathcal{H}_0\). In fact, if \(h\) is in \(H\) we have \(a_t(G_t a_t f, h) = (a_t f, h) = a_t(f, h)\) so that \(G_t a_t f = f\).

Let \(k\) be an integer \(> 0\). We shall prove that \(G_t^k\) has a kernel \(g_t^{(k)}(x, y)\) so that
\[
\langle G_t^k f, f' \rangle = \int_{S \times S} g_t^{(k)}(x, y) f(x) f'(y) \, dx \, dy
\]
\(^2\) In [3] the operator \(a\) is denoted by \(q\). Introducing the operator \(R_t\) defined in the proof of Lemma 4.1 (l. c. p. 69), we may write \(q_t(f, f)\) as \(p_t(f + R_t f, f)\). Hence we get
\[
p_t(f, f)(1 - |R_t|) = |q_t(f, f)| \leq p_t(f, f)(1 + |R_t|),
\]
the norm \(|R_t|\) being defined in l. c. p. 69. Now \(|R_t|\) tends to zero with \(1/t\), and hence Theorem 2.2 of l. c. proves that the formula (0.3) above is true.
when $f, f' \in H(S)$. The kernel is infinitely differentiable when $x \neq y$ and has the singularity to be expected when $|x - y|$ is small. If $2km > n$ it is continuous and satisfies

$$
\lim_{t \to \infty} t^{k-n} g_t^{(k)}(x, y) = \delta_{xy} (2\pi)^{-n} \int (a_0(x, \xi) + 1)^{-k} d\xi
$$

where $\delta_{xy} = 0$ when $x \neq y$ and $\delta_{xx} = 1$ and $v = n/(2m)$. If $a$ is formally self-adjoint, that is, if $(af, f)$ is real for all $f \in H$, then $G_t$ is also self-adjoint,

$$
\text{tr } G_t^k = \int_S g_t^{(k)}(x, x) \, dx < \infty
$$

if $2mk > n$, and

$$
\lim_{t \to \infty} t^{k-n} \text{tr } G_t^k = (2\pi)^{-n} \int \int_S (a_0(x, \xi) + 1)^{-k} d\xi.
$$

Moreover, if $a$ is self-adjoint, there exists a set $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of every $G_t$ with eigenvalues

$$
(\lambda_1 + t)^{-1}, (\lambda_2 + t)^{-1}, \ldots \quad (\lambda_1 \leq \lambda_2 \leq \ldots).
$$

The eigenfunctions form a complete orthonormal system in $S_0$. If $2mk > n$ we have

$$
g_t^{(k)}(x, y) = \sum \overline{\varphi_j(x)} \varphi_j(y) (\lambda_j + t)^{-k}
$$

and

$$
\text{tr } G_t^k = \sum (\lambda_j + t)^{-k}.
$$

We can now apply the method of Carleman [2] to deduce some asymptotic formulas for the eigenfunctions and eigenvalues. Our last two formulas combined with (0.4) and (0.5) and a Tauberian theorem of Hardy and Littlewood [4] in the formulation of Pleijel [9] show in fact that

$$
N(t) = \sum_{\lambda_j \leq t} 1 = (2\pi)^{-n} w_a(S) t^{n/(2m)} (1 + o(1))
$$

and

$$
\lim_{N \to \infty} N^{-1} \sum_{j=1}^N \varphi_j(x) \varphi_j(y) = \delta_{xy} w_a(x)/w_a(S),
$$

where

$$
w_a(x) = \int a_0(x, \xi) < 1 d\xi \quad \text{and} \quad w_a(S) = \int_S w_a(x) \, dx.
$$

These formulas are well known in various special cases. They were stated by the author [3] with indications of a proof which works when $2m > n$.

---

3 Essentially this proof has been published in detail by Browder [1].
In this paper we shall follow another line of attack to obtain the key formulas (0.4) and (0.5).

— Our results combined with those announced by Keldysh [7] prove that the asymptotic formula (0.6) is true also when $\alpha$ is not self-adjoint, provided that we replace $\lambda_j$ by $\Re \lambda_j$.

1. Two lemmas. We shall use the theory of generalized Fourier transforms (Schwartz [10]). Let $F = F(x) = F(x_1, \ldots, x_n)$ be an infinitely differentiable function on real $n$-space vanishing outside a compact set. It has a Fourier transform $f$ given by

$$f(\xi) = \int e^{ix\xi} F(x) \, dx,$$

where $x\xi = x_1\xi_1 + \ldots$. It is well known that $f(\xi) = O((1+|\xi|)^{-N})$ for every $N$ ($|\xi| = (\xi_1^2 + \ldots)^{1/2}$). The inverse formula reads

$$F(x) = (2\pi)^{-n} \int e^{-ix\xi} f(\xi) \, d\xi.$$

Let $a(\xi)$ be a locally integrable function which is $O(1+|\xi|^N)$ for some $N > 0$. Then

$$A(F) = (2\pi)^{-n} \int a(\xi) \overline{f(\xi)} \, d\xi$$

defines an antilinear functional $A$ of $F$ called the generalized inverse Fourier transform of $a$. If $(2\pi)^{-n} \int |a(\xi)| \, d\xi = c < \infty$, the function

$$A(x) = (2\pi)^{-n} \int e^{-ix\xi} a(\xi) \, d\xi$$

is continuous and defined for all $x$, and $|A(x)| \leq c$. In this case

$$A(F) = \int A(x) \overline{F(x)} \, dx.$$

More generally, we say that $A$ is a function $A(x)$ in a region $R$ if there exists a locally integrable function $A(x)$ in $R$ for which this equation holds when $F$ vanishes outside a compact set in $R$.

Returning to the general case, let $D^\alpha$ be defined by (0.1). The derivative $D^\alpha A$ of $A$ is defined by

$$D^\alpha A(F) = (2\pi)^{-n} \int \xi^\alpha a(\xi) \overline{f(\xi)} \, d\xi,$$

where $\xi^\alpha$ is defined in (0.2). The product $\lambda A$ of $A$ and a polynomial $\lambda(x)$ is defined by

$$\lambda A(F) = (2\pi)^{-n} \int a(\xi) \lambda(D_\xi)f(\xi) \, d\xi,$$

where $\lambda(D_\xi) = \lambda(i\partial/\partial\xi_1, \ldots, i\partial/\partial\xi_n)$.
Among the results of Schwartz we quote the following ones. If the product of $A$ and a polynomial $\lambda$ which is not zero in a region $R$ is a function $B(x)$ in $R$, then $A$ is the function $B(x)/\lambda(x)$ in $R$. If all the derivatives of $A$ are functions in a region $R$, then $A$ is infinitely differentiable there and its ordinary derivatives $D^s A(x)$ are related to $D^s A(F)$ by the formula

$$D^s A(F) = \int D^s A(x) \overline{F(x)} \, dx \quad (F \in H(R)).$$

We are now in a position to prove the following lemma.

**Lemma 1.** Let $p(\xi)$ be a polynomial of degree $\mu$ whose coefficients are majorized by a number $c_1$, and suppose that $|p(\xi)| \geq c_2 (1 + |\xi|^\nu)$. Then the generalized inverse Fourier transform of $1/p(\xi)$ is an infinitely differentiable function $P(x)$ in the region $x = 0$ satisfying

$$|D^s P(x)| \leq C e_{|x|}(x) \left(1 + |x|^N\right)^{-1}, \quad \left\{ \begin{array}{ll}
e_{|x|}(x) = 1 & \text{when } \mu - |x| - n > 0, \\
e_{|x|}(x) = |x|^{\mu - |x| - n - \varepsilon} & \text{when } \mu - |x| - n \leq 0. \end{array} \right.$$ 

Here $N \geq 0$ and $1 > \varepsilon > 0$ are arbitrary, and the number $C$ depends on $c_1, c_2, |\xi|, N,$ and $\varepsilon$, but is otherwise independent of the polynomial $p$.

**Proof.** Let $\lambda$ be a polynomial. Then

$$\lambda D^s P(F) = (2\pi)^{-n} \int \xi^{s} p^{-1}(\xi) \lambda(D_\xi) \overline{f(\xi)} \, d\xi.$$ 

By virtue of the properties of $p$, we may integrate by parts and get

$$\lambda D^s P(F) = (2\pi)^{-n} \int \overline{f(\xi)} \lambda(-D_\xi)(\xi^{s} p^{-1}(\xi)) \, d\xi.$$ 

Let $\lambda$ be homogeneous of degree $k \leq N$ and let its coefficients be $\leq 1$ in absolute value. Let $C$ denote a suitable number, not always the same, but depending only on $|\xi|, c_1, c_2, N$, and later also on $\varepsilon$. It is clear that

$$|\lambda(-D_\xi)(\xi^{s} p^{-1}(\xi))| \leq C (1 + |\xi|)^{|\xi| - \mu - k}.$$ 

Hence, if $|\xi| - \mu < -n$ and $\lambda$ has the properties stated, it follows that $\lambda D^s P$ is a function majorized by $C$. This proves the first half of the lemma.

Next consider the case $|\xi| - \mu \geq -n$. Supposing that the coefficients of $\lambda$ are $\leq 1$ in absolute value, we let $\lambda$ be homogeneous of degree $k + n + |\xi| - \mu$ where $1 \leq k \leq N + 1$. Then

$$|\lambda(-D_\xi)(\xi^{s} p^{-1}(\xi))| \leq C (1 + |\xi|)^{-n - k},$$

and if $\lambda D^s P$ is the function

$$(2\pi)^{-n} \int e^{-ix\xi} \lambda(-D_\xi)(\xi^{s} p^{-1}(\xi)) \, d\xi.$$ 

---

4 Actually, it is analytic (Schwartz [10], F. John [9]).
This function vanishes when \( x = 0 \), the integrand being a sum of exact differentials, and hence we may replace \( e^{-ix\xi} \) by \( e^{-ix\xi} - 1 \) whose absolute value is less than \( C|x|^{1-\varepsilon} |\xi|^{1-\varepsilon} \), so that \( \lambda D^x P \) is majorized by \( C|x|^{1-\varepsilon} \).

Hence, if \( \lambda \) is a polynomial with the properties stated above, then \( \lambda D^x P \) is a function bounded by \( C|x|^{1-\varepsilon} \). The second half of the lemma follows.

The function \( P(x) \) is a fundamental solution of the differential operator \( p(-D_x) \) in the sense that

\[
\int P(x-y) p(D_y) F(y) \, dy = F(x) .
\]

We shall express the connection between \( P \) and \( p \) by the symbolic notation

\[
P(x) \sim (2\pi)^{-n} \int e^{-ix\xi} p^{-1}(\xi) \, d\xi .
\]

It is clear that linear coordinate transformations preserve the sense of this formula.

Obviously, \( e_{\mu}(x) \) is majorized by \( e_{|x|}(x) \) if \( |\alpha| \leq |x| \) and \( x \) is bounded. For large \( |x| \), \( |D^x P(x)| \leq C|x|^{-N} \) for arbitrary \( N \), and hence we obtain from Lemma 1 the supplementary

**Lemma 2.** Under the hypotheses of Lemma 1, the following estimates hold:

\[
|D^x P(x)| \leq C|x|^{1-\varepsilon-n}(1+|x|^N)^{-1} \quad (|\alpha| < \mu)
\]

and

\[
|D^x P(x)| \leq C|x|^{-\varepsilon-n}(1+|x|^N)^{-1} \quad (|\alpha| \leq \mu) .
\]

Here \( N \geq 0 \) and \( 1 > \varepsilon > 0 \) are arbitrary, and the number \( C \) depends on \( c_1, c_2, N, \) and \( \varepsilon \), but is otherwise independent of the polynomial \( p \).

2. Estimates of certain fundamental solutions. Let \( \tau \) be a large positive real parameter and consider a differential operator of order \( \mu \)

\[
b = b(\tau, x, D_x) = \sum_{|\alpha| \leq \mu} b_\alpha(\tau, x) \tau^{-|\alpha|} D_x^\alpha ,
\]

where \( b_\alpha(\tau, x) \) is a polynomial in \( \tau^{-1} \) whose coefficients are infinitely differentiable functions in an open region \( T \). It is assumed that the following polynomial in \( \xi \)

\[
b_0(\tau, x, \xi) = b(\tau, x, \tau \xi) = \sum b_\alpha(\tau, x) \xi^\alpha
\]

has the property that
(2.1) \[ b_0^{-1}(\infty, x, \xi) = O(1) (1 + |\xi|^n)^{-1}, \]

uniformly on compact subsets of \( T \).

The algebraic adjoint \( b^* \) of \( b \) is defined by the identity
\[
\int b^* f(x) f'(x) \, dx = \int f(x) b f'(x) \, dx, \quad (f, f' \in H(T)).
\]

It is readily seen to have the same form as \( b \) itself,
\[
b^*(\tau, x, D_x) = \sum_{|\alpha| \leq \mu} b_\alpha^*(\tau, x) \tau^{-|\alpha|} D_x^\alpha,
\]
where \( b_\alpha^*(\tau, x) \) is a polynomial in \( \tau^{-1} \) and \( b_0^*(\infty, x, \xi) \) satisfies (2.1) since, in fact, we have
\[
(2.2) \quad b_0^*(\infty, x, \xi) = b_0(\infty, x, -\xi).
\]

Let \( U \) be an arbitrary open subset of \( T \) whose closure \( \bar{U} \) is contained in \( T \). When \( \tau \) is large enough we are going to construct a fundamental solution \( \Gamma \) of the differential operator \( b \), that is, a function \( \Gamma(\tau, x, z) \) defined on \( U \times U \) and having the property that
\[
\int \Gamma(\tau, z, x) b^*(\tau, x, D)f(x) \, dx = f(z) \quad (f \in H(U)).
\]
The point \( z \) is called the pole of \( \Gamma \). We shall establish the estimates
\[
(2.3) \quad \Gamma(\tau, z, x) = O(1) \tau^n e_0(\tau(x-z)) (1 + |\tau(x-z)|^N)^{-1}
\]
(where \( e_0(y) = |y|^{\mu-n-\varepsilon} \) when \( \mu - n \leq 0 \), and \( e_0(y) = 1 \) otherwise) and
\[
(2.4) \quad D_x^\alpha \Gamma(\tau, z, x) = O(1) \tau^{|\alpha|+1-\varepsilon} |x-z|^{1-\varepsilon-n} (1 + |\tau(x-z)|^N)^{-1} \quad (|\alpha| < \mu).
\]

In these formulas \( N \geq 0 \) and \( 1 > \varepsilon > 0 \) are arbitrary, and the estimate \( O(1) \) for large \( \tau \) is uniform in \( U \times U \). We shall also prove that, if \( \mu > n \), then
\[
(2.5) \quad \lim_{\tau \to \infty} \tau^{-n} \Gamma(\tau, x, x) = (2\pi)^{-n} \int d\xi / b_0(\infty, x, \xi),
\]
uniformly on the diagonal \( \Delta(U \times U) \) of \( U \times U \). Precisely the same estimates hold for a similarly constructed fundamental solution \( \Gamma^*(\tau, x, z) \) with pole in \( z \) defined in \( U \times U \) for large \( \tau \) and satisfying\(^5\)
\[
\int \Gamma^*(\tau, x, z) b(\tau, x, D)f(x) \, dx = f(z)
\]
when \( f \) is in \( H(U) \). For future reference we write them down:
\[
(2.6) \quad \Gamma^*(\tau, x, z) = O(1) \tau^n e_0(\tau(x-z)) (1 + |\tau(x-z)|^N)^{-1},
\]

\(^5\) It is convenient to change the parts played by the last two variables in \( \Gamma \) and \( \Gamma^* \).
(2.7) \( D_x^* \Gamma^*(\tau, x, z) = O(1) \tau^{1+1-\epsilon} |x-z|^{1-\epsilon-n} (1 + |\tau(x-z)|^N)^{-1} \quad (|\tau| < \mu) \),

(2.8) \[ \lim_{\tau \to \infty} \tau^{-n} \Gamma^*(\tau, x, x) = (2\pi)^{-n} \int d\xi / b_0^*(\infty, x, \xi). \]

By virtue of (2.2), the right sides of (2.5) and (2.8) are the same.

We shall use the current parametrix method of Hilbert [5] and E. E. Levi [8]. The parametrix is a fundamental solution \( B(\tau, z, x) \) of the differential operator \( b(\tau, z, -D_x) \) with its pole at \( z \). Put

(2.9) \[ B(\tau, z, x) \sim (2\pi)^{-n} \int e^{-i(x-x')\xi} b^{-1}(\tau, z, \xi) d\xi \sim \tau^n (2\pi)^{-n} \int e^{-ix'\xi} b^{-1}(\tau, z, \tau\xi) d\xi \]

where \( x' = \tau(x-z) \). By virtue of (2.1) the polynomial \( b(\tau, z, \tau\xi) \) never vanishes if \( \tau \) is large enough, so that

\[ \int B(\tau, z, x) b(\tau, z, -D_x)f(x) \, dx = f(z) \quad (f \in H(T)). \]

Also, by virtue of (2.1), the polynomial \( b(\tau, z, \tau\xi) \) satisfies the requirements of Lemma 1 on compact subsets of \( T \). Hence by Lemma 2,

\[ D_x^* B(\tau, z, x) = O(1) \tau^n |x'|^{1-\epsilon-n} (1 + |x'|^N)^{-1} \quad (|\tau| < \mu), \]

so that

(2.10) \[ D_x^* B(\tau, z, x) = O(1) \tau^{1+1-\epsilon} |x-z|^{1-\epsilon-n} (1 + |\tau(x-z)|^N)^{-1} \quad (|\tau| < \mu). \]

Similarly

(2.11) \[ D_x^* B(\tau, z, x) = O(1) \tau^{1-\epsilon} |x-z|^{1-\epsilon-n} (1 + |\tau(x-z)|^N)^{-1} \quad (|\tau| \leq \mu). \]

Both of these estimates are valid for arbitrary \( N \geq 0 \) and \( 1 > \epsilon > 0 \), uniformly for sufficiently large \( \tau \), all \( x \) and all \( z \) on any compact part of \( T \). Let us put

\[ \beta(\tau, z, x) = (b(\tau, z, D_x) - b(\tau, x, D_x)) B(\tau, z, x), \]

and let \( u(\tau, z, x) \) be a solution of the integral equation

(2.12) \[ u(\tau, z, x) - \int_U u(\tau, z, y) \beta(\tau, y, x) \, dy = \beta(\tau, z, x). \]

Then

(2.13) \[ \Gamma(\tau, z, x) = B(\tau, z, x) + \int_U u(\tau, z, y) B(\tau, y, x) \, dy \]

is a fundamental solution in \( U \times U \) (see F. John [6]). We shall investigate the possibility of solving (2.12). Writing \( \beta(\tau, z, x) \) more explicitly as

\[ \Sigma(b_\alpha(\tau, z) - b_\alpha(\tau, x)) \tau^{-|\alpha|} D_x^* B(\tau, z, x) \]

and using (2.10) and (2.11), we get
\( \beta(\tau, z, x) = O(1) \tau^{-\varepsilon} |x-z|^{1-\varepsilon-n} (1+|\tau(x-z)|^N)^{-1} \)

for arbitrary \( N \) and \( \varepsilon \), uniformly when \( \tau \) is large and \( x, z \) belongs to a compact part of \( T \times T \). Hence the Neumann series of (2.12), namely

\[
\beta(\tau, z, x) + \int \beta(\tau, z, y)\beta(\tau, y, x)dy + \ldots,
\]

is majorized by

\[
(1+|\tau(x-z)|^N)^{-1}\left\{ C\tau^{-\varepsilon} |x-z|^{1-\varepsilon-n} + (C\tau^{-\varepsilon})^2 \int_U |z-y|^{1-\varepsilon-n} |y-x|^{1-\varepsilon-n}dy + \ldots \right\},
\]

where \( C \) is a constant. We have here used the simple inequality

\[
(1+|\tau(z-y)|^N)^{-1}(1+|\tau(y-x)|^N)^{-1} \leq (1+|\tau(z-x)|^N)^{-1}.
\]

Hence, if \( \tau \) is large enough, the integral equation may be solved by its Neumann series, and we get

(2.14) \( u(\tau, z, x) = O(1) \tau^{-\varepsilon} |x-z|^{1-\varepsilon-n} (1+|\tau(x-z)|^N)^{-1} \)

for large \( \tau \), uniformly on \( U \times U \). This estimate together with (2.13) proves the desired estimate (2.7), because, as it stands, we may clearly differentiate (2.13) with respect to \( x \) less than \( \mu \) times. From (2.9) and Lemma 1 follows

(2.15) \( B(\tau, z, x) = O(1)\tau^n e_0(\tau(x-z)) (1+|\tau(x-z)|^N)^{-1} \),

which together with (2.12) and (2.13) gives (2.6).

It remains to prove (2.8). From (2.9) and the properties of \( b \) it follows that

\[
\lim_{\tau \to \infty} \tau^{-n} B(\tau, x, x) = (2\pi)^{-n} \int d\xi/b_0(\infty, x, \xi),
\]

uniformly on any compact subset of \( \Delta(T \times T) \). Now if \( \mu > n \), (2.15) reads

\[
B(\tau, z, x) = O(1)\tau^n (1+|\tau(x-z)|^N)^{-1}.
\]

Combining this with the estimate (2.14) of \( u \), we see that \( \tau^{-n} \) times the integral in (2.13) is uniformly small on \( U \times U \) if \( \tau \) is large enough. Hence (2.8) follows. For \( I^* \) the construction and the proofs are the same.

At last we remark that according to the results of F. John, \( \Gamma(\tau, z, x) \) is infinitely differentiable in \( U \times U \) if \( x \neq z \), and if \( f \) is in \( H(U) \) then

(2.16) \( \Gamma f(x) = \int \Gamma(\tau, z, x)f(z)dz \)

is infinitely differentiable in \( U \). Moreover,

(2.17) \( b(\tau, x, D_x)\Gamma(\tau, z, x) = 0 \quad (x \neq z) \),
and
\begin{equation}
(2.18) \quad b(\tau, x, D_x)\varGamma f(x) = f(x).
\end{equation}

Analogous results hold for $\varGamma^*(\tau, x, z)$ and
\[ \varGamma^* f(x) = \int \varGamma^*(\tau, x, z) f(z) dz. \]

3. Estimates of a certain kernel. Let us use the notations of the preceding section, and let $\mathcal{S}_0(T)$ be the set of all square integrable functions on $T$. Assume that a bilinear form
\[ C(f, g) = C(\tau, f, g) \]
is given on $\mathcal{S}_0 \times \mathcal{S}_0$ and that it is uniformly bounded for large $\tau$ and satisfies the identity
\begin{equation}
(3.1) \quad C(bf, g) = C(f, b^*g) = \int f(x) g(x) dx
\end{equation}
on $H(T) \times H(T)$. Then we shall prove that $C$ has a kernel $c(\tau, x, y)$,
\[ C(f, g) = \int_{T \times T} c(\tau, x, y) f(x) g(y) dy \quad \text{for} \quad (f, g \in H(T)), \]
which is infinitely differentiable when $x \neq y$ and satisfies
\begin{equation}
(3.2) \quad c(\tau, x, y) = O(1) \tau^\varepsilon e_0(\tau(x-y))(1 + |\tau(x-y)|^N)^{-1},
\end{equation}
where as usual
\[ e_0(z) = \begin{cases} |z|^{\mu-n-\varepsilon} & \text{when } \mu-n \leq 0, \\ 1 & \text{when } \mu-n > 0. \end{cases} \]
The estimate $O(1)$ is uniform for large $\tau$ and compact subsets of $T \times T$, but it depends on the numbers $N$ and $\varepsilon$ which may be chosen except for the conditions $N \geq 0$ and $1 > \varepsilon > 0$. When $\mu-n > 0$,
\begin{equation}
(3.3) \quad \lim_{\tau \to \infty} \tau^\varepsilon c(\tau, x, y) = (2\pi)^{-n} \delta_{xy} \int_{\mathbb{R}^n} d\xi / b_0(\infty, x, \xi),
\end{equation}
uniformly on compact subsets of $T \times T$, the symbol $\delta_{xy}$ denoting 0 when $x \neq y$ and 1 otherwise.

Let $V$ be an arbitrary open subset of $T$ whose closure $\bar{V}$ is contained in $T$, and choose three larger telescoping open subsets $V_1, V_2$ and $V_3$ such that $V \subset \bar{V} \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset V_3 \subset \bar{V}_3 \subset T$. Construct fundamental solutions $\varGamma$ and $\varGamma^*$ in $U = V_3$ satisfying (2.3) to (2.8). Let $\psi$ be in $H(V_3)$ and let it be 1 on $V_2$. We want to prove that
\begin{equation}
(3.4) \quad q(\tau, x, x^*) = \varGamma^*(\tau, x, x^*) - \varGamma(\tau, x, x^*)
= \int_{V_3-V_2} \varGamma(\tau, x, z) b^*(\tau, z, D_z)(\psi(z) \varGamma^*(\tau, z, x)) dz.
\end{equation}
when \( x \) and \( x^* \) are in \( V_2 \). This formula in another form is due to F. John [6], but for the convenience of the reader we give a proof.

First let \( x = x^* \) and choose three open telescoping neighborhoods \( W, W_1 \) and \( W_2 \) of \( x^* \) such that \( W \subset W_1 \subset W_1 \subset W_2 \subset W_2 \subset V_2 \) and \( W_2 \) does not contain \( x \) (see the figure).

The shaded rings are those where the corresponding functions are \( \pm 0 \) and \( \pm 1 \). Each function equals 1 inside its ring and 0 outside it.

Let \( \varphi \in H(W_1) \) be 1 on \( W \) and \( \varphi_1 \in H(W_2) \) be 1 on \( W_1 \). Put \( f(z) = \Gamma(\tau, x, z) \) and \( f^*(z) = \Gamma^*(\tau, z, x^*) \). Then, by the properties of \( \Gamma \) and \( \Gamma^* \),

\[
[f, b^*(\psi f^*)] = f^*(x) \quad ([f, g] = \int f(x)g(x)\,dx).
\]

Since \( b^*f^* = 0 \) except at the point \( z = x \), the left side can be written as

\[
[f, b^*(\psi f^*)] + [f, b^*(1-\varphi)f^*].
\]

But

\[
[f, b^*(1-\varphi)f^*] = [\varphi_1 f, b^*(1-\varphi)f^*] = [b\varphi_1 f, (1-\varphi)f^*] = [b\varphi_1 f, f^*] = f(x^*),
\]

and hence

\[
f^*(x) - f(x^*) = [f, b^*(\psi f^*)],
\]

which is the formula (3.4). By continuity, the formula just proved is valid also when \( x = x^* \). This shows that \( \varphi \) is infinitely differentiable on \( V_2 \times V_2 \). Put

\[
\Gamma^*(\tau, f, g) = \int \Gamma^*(\tau, x, x^*)f(x)g(x^*)\,dx\,dx^*
\]

and

\[
C'(f, g) = C(f, g) - \Gamma^*(\tau, f, g).
\]

By virtue of (2.4) with \(|\alpha| = 0\), the norm
\[ |C'| = \sup |C'(f, g)| |f|^{-1} |g|^{-1} \quad (|f|^2 = \int_{V_2} \|f(x)\|^2 \, dx; \quad f, g \in \mathcal{S}_0(V_2)) \]

of the bilinear form \( C' \) is \( O(1) \tau^{-1-\varepsilon} \). Moreover, by the properties of \( \Gamma^* \),

\[ C'(bf, g) = [f, g] - [f, g] = 0, \]

and if \( \Gamma(\tau, f, g) \) and \( \varrho(\tau, f, g) \) are defined in analogy to the above definition of \( \Gamma^*(\tau, f, g) \),

\[ C'(f, b^*g) = C(f, b^*g) - \Gamma(\tau, f, b^*g) - \varrho(\tau, f, b^*g) = -\varrho(\tau, f, b^*g) \]
on \( H(V_2) \times H(V_2) \). Put

\begin{align}
(3.5) \quad p(x, z) &= b(\tau, z, D_z)(\Gamma(\tau, x, z)(1 - \varphi(z))) \\
(3.6) \quad p^*(z, x^*) &= b(\tau, z, D_z)(\Gamma^*(\tau, z, x^*)(1 - \varphi(z))) ,
\end{align}

where \( \varphi \in H(V_2) \) and equals 1 on \( V_1 \). For given \( x \) and \( x^* \) in \( V_1 \), the functions \( p(x, z) \) and \( p^*(z, x^*) \) vanish except on the ring \( V_2 - V_1 \). Put further

\[ c'(\tau, x, x^*) = C'(p(x, .), p^*(., x^*)) . \]

By the continuity of \( p \), \( p^* \) and \( C' \),

\[ c'(\tau, f, g) = \int c'(\tau, x, x^*)f(x)g(x^*) \, dx \, dx^* = C'(f - b\varphi \Gamma f, g - b^*\varphi \Gamma^* g) , \]

where \( f \) and \( g \) are in \( H(V) \),

\[ \Gamma f(z) = \int \Gamma(\tau, x, z)f(x) \, dx , \]

and

\[ \Gamma^* g(z) = \int \Gamma^*(\tau, z, x^*)g(x^*) \, dx^* . \]

By the properties of \( C' \),

\[ c'(\tau, f, g) = C'(f, g) + \varrho(\tau, f, b^*\varphi \Gamma^* g) . \]

Now

\[ b^*\varphi \Gamma^* g(z) = b^*(\tau, z, D_z)(\varphi(z) \int \Gamma^*(\tau, z, x^*)g(x^*) \, dx^*) \]

\[ = g(z) + \int b^*(\tau, z, D_z)(\varphi(z) \Gamma^*(\tau, z, x^*))g(x^*) \, dx^* , \]

where the kernel

\begin{equation}
(3.8) \quad \sigma(\tau, z, x^*) = b^*(\tau, z, D_z)(\varphi(z) \Gamma^*(\tau, z, x^*))
\end{equation}

is different from zero only when \( z \) is in \( V_2 - V_1 \). Hence

\[ C(f, g) = \Gamma^*(\tau, f, g) + c'(\tau, f, g) + \varrho(\tau, f, g) + \lambda(\tau, f, g) , \]

where

\begin{equation}
(3.9) \quad \lambda(\tau, x, x^*) = \int_{V_2 - V_1} \varrho(\tau, x, z)\sigma(\tau, z, x^*) \, dz .
\end{equation}
Consider now the functions \( p(x, z) \) and \( p^*(z, x^*) \). Since

\[
\begin{align*}
b(\tau, z, D_z) \Gamma(\tau, x, z) &= 0 \\
b^*(\tau, z, D_z) \Gamma^*(\tau, z, x^*) &= 0
\end{align*}
\]

when \( x \neq z \), and

\[
\begin{align*}
b^*(\tau, z, D_z) \Gamma^*(\tau, x, z^*) &= 0
\end{align*}
\]

when \( x^* \neq z \), only derivatives of orders \( < \mu \) of \( \Gamma \) and \( \Gamma^* \) really enter into \( p \) and \( p^* \), respectively. Hence by (2.4)

\[
p(x, z) = O(1) \tau^{1-\varepsilon} r^{1-\varepsilon-n} (1 + |\tau r|^N)^{-1},
\]

where \( r \) is the distance from \( x \) to \( V_1 \), and analogously,

\[
p(z, x) = O(1) \tau^{1-\varepsilon} r^*^{1-\varepsilon-n} (1 + |\tau r^*|^N)^{-1},
\]

where \( r^* \) is the distance from \( x^* \) to \( V_1 \). Hence

\[
(3.10) \quad p(x, z) = O(1) \tau^{1-\varepsilon-N}
\]

and

\[
(3.11) \quad p^*(z, x^*) = O(1) \tau^{1-\varepsilon-N}
\]

uniformly on \( V \times (V_2 - V_1) \) and \( (V_2 - V_1) \times V \), respectively. In a similar fashion we infer from (3.4) that

\[
(3.12) \quad q(\tau, x, z) = O(1) \tau^{2(1-\varepsilon)-N}
\]

uniformly on \( V_1 \times V_2 \), and

\[
(3.13) \quad q(\tau, x, x^*) = O(1) \tau^{2(1-\varepsilon-N)}
\]

uniformly on \( V_1 \times V_1 \). Further,

\[
\sigma(\tau, z, x) = O(1) \tau^{1-\varepsilon-N}
\]

uniformly on \( V_2 \times V_1 \). Hence by (3.9),

\[
(3.14) \quad \lambda(\tau, x, x^*) = O(1) \tau^{3(1-\varepsilon)-2N}
\]

uniformly on \( V_1 \times V_1 \). Combining (3.10) and (3.11) with the estimate \( O(1) \tau^{1-\varepsilon} \) of \(|C'|\), we get

\[
(3.15) \quad c'(\tau, x, x^*) = O(1) \tau^{3(1-\varepsilon)-2N}
\]

uniformly on \( V \times V \). Combining (3.13), (3.14), and (3.15) it follows that \( C(f, g) \) has a kernel

\[
c(\tau, x, x^*) = \Gamma^*(\tau, x, x^*) + O(1) \tau^{3(1-\varepsilon)-2N},
\]

\( O(1) \) being uniform on \( V \times V \). Hence the desired formulas (3.2) and (3.3) follow from (2.6) and (2.8), respectively.
4. Estimates of Green’s function. In the introduction, Green’s transformation $G_t$ was defined for sufficiently large $t$ by

\[(f, f') = a_t(G_t f, f'),\]

where $f \in \mathcal{H}_0(S)$ and $G_t f, f' \in \mathcal{H}(S)$. It is clear that $\|f\|^2 \geq t |f|^2$ and hence by (0.3),

\[tc^{-1}|G_t f|^2 \leq c^{-1}\|G_t f\|^2 \leq |a_t(G_t f, G_t f)| \leq |f| |G_t f|,
\]

so that

\[|G_t f| \leq ct^{-1}|f|.
\]

Considering $G_t$ as an operator from $\mathcal{H}_0$ to $\mathcal{H}_0$ we therefore have

\[|G_t| \leq ct^{-1}.
\]

We have already shown that if $f$ is in $H(S)$ then

\[(4.2) \quad G_t a_t f = f.
\]

Let $a_t^*$ be the complex conjugate adjoint of $a_t$ defined by

\[(a_t f, f') = (f, a_t^* f') \quad (f, f' \in H(S)).
\]

If $f \in \mathcal{H}(S)$ and $f' \in H(S)$, we then have

\[(4.3) \quad (G_t f, a_t^* f') = a_t(G_t f, f') = (f, f').
\]

Let $k$ be a positive integer and put

\[C(f, f') = (G_t^k f, \bar{f'} ) t^k
\]

and $b = t^{-k} a_t^k$. Then $C(f, f')$ is bilinear and bounded,

\[|C(f, f')| \leq c |f| |f'|,
\]

and by virtue of (4.1),

\[C(b f, f') = (G_t^k a_t^k f, \bar{f'}) = (f, \bar{f'}) \quad (f, f' \in H(S)).
\]

If $b^*$ is the algebraic adjoint of $b$, it follows from (4.3) that

\[C(f, b^* f') = (G_t^k f, a_t^k \bar{f'}) = (f, \bar{f'}) \quad (f, f' \in H(S)).
\]

Moreover, putting $t^{2m} = t$, it is clear that

\[b = b(\tau, x, D_x) = \sum b_\alpha(\tau, x) \tau^{-|\alpha|} D_x^\alpha
\]

has the property (2.1). Hence, applying the results of the preceding section we see that $G_t^k$ has a continuous kernel $g_t^{(k)}(x, y)$ with the property

\[(4.4) \quad \lim_{t \to \infty} t^{k-v} g_t^{(k)}(x, y) = \delta_{xy} (2\pi)^{-n} \int (a_0(x, \xi) + 1)^{-k} d\xi,
\]

provided that $2mk > n \quad (v = n/(2m)).$
If \( a_t \) is self-adjoint, that is, if \( a^*_t = a_t \), then \( a_t(f, f) = (a_t f, f) = (f, a^*_t f) \) is real when \( f \in \mathfrak{H}(S) \). Hence \( a_t(f, f) \) is real when \( f \in \mathfrak{H}(S) \) and consequently \( (G_t f, f) = a_t(G_t f, G_t f) \) is real when \( f \in \mathfrak{H}_0 \). By virtue of (0.3) also \( (G_t f, f) \geq 0 \) when \( t \) is large enough, and hence \( G_t \) is a self-adjoint positive transformation. But then \( g_t^{(k)}(x, x) \geq 0 \), and Fatou’s theorem gives

\[
(4.5) \quad \lim_{x \to -\infty} t^{k-n} \int_S g_t^{(k)}(x, x) d\xi \geq (2\pi)^{-n} \int_S \left[ (a_0(x, x) + 1)^{-k} d\xi \right].
\]

It remains to prove the converse inequality.

Let \( \hat{S} \) be an open set in \( T \) containing the closure of \( S \), and let \( \hat{G}_t \) be Green’s transformation corresponding to \( \hat{S} \). By virtue of (4.1) we have

\[
(G_t f, f) = a_t(G_t f, G_t f) = \sup \left| (f, g)^2/a_t(g, g) \right| \quad (g \in \mathfrak{H}(S)).
\]

Since \( \mathfrak{H}(\hat{S}) \) contains \( \mathfrak{H}(S) \) this means that

\[
(G_t f, f) \leq (\hat{G}_t f, f) = (E \hat{G}_t f, f) \quad (f \in \mathfrak{H}_0(S)),
\]

where \( E \) is the projection of \( \mathfrak{H}_0(\hat{S}) \) upon \( \mathfrak{H}_0(S) \). Hence all the eigenvalues of \( G_t \) taken in descending order are less than or equal to the corresponding eigenvalues of the restriction \( G_t = E \hat{G}_t E \) of \( \hat{G}_t \) to \( \mathfrak{H}_0(S) \). Hence

\[
tr G_t^k \leq tr \Gamma_t^k
\]

for all \( k \). We want to obtain an estimate for the right side.

To begin with, it is clear that the bilinear forms \( (\hat{G}_t f, \hat{f}) t^k \) and \( (\Gamma_t f, f) t^k \) both satisfy the requirements of the preceding section with respect to the differential operator \( b = t^{-k}a_t \) and the regions \( \hat{S} \) and \( S \), respectively. In particular, \( \Gamma_t^k \) has a kernel \( \gamma_t^{(k)}(x, y) \) which is continuous when \( 2mk > n \) and satisfies

\[
(4.6) \quad \lim_{y \to \infty} t^{k-n} \gamma_t^{(k)}(x, x) = (2\pi)^{-n} \int_S \left[ (a_0(x, \xi) + 1)^{-k} d\xi \right].
\]

The kernel \( \gamma_t(x, y) \) of \( \Gamma_t \) is the restriction of the kernel \( \hat{g}_t(x, y) \) of \( \hat{G}_t \) to \( S \), and hence from (3.2) we get the estimate

\[
\gamma_t(x, y) = t^{-1} O(1) e(\tau(x-y))(1 + |\tau(x-y)|^N)^{-1}
\]

with arbitrary \( N \geq 0 \) and \( 1 > \epsilon > 0 \), uniformly on \( S \times S \) for large \( t \) (\( \tau^{2m} = t \)). Here \( e(x) = 1 \) if \( 2m - n > 0 \), and \( e(x) = |x|^{2m-n-\epsilon} \) otherwise. By virtue of this estimate and Fubini’s theorem, the kernel of \( \Gamma_t^k \) is

\[
\gamma_t^{(k)}(x, y) = \int_{S^{k-1}} \gamma_t(x, y_1) \gamma_t(y_1, y_2) \cdots \gamma_t(y_{k-1}, y) dy_1 \cdots dy_{k-1}.
\]

Putting \( y = x \) and \( y_j = x + \tau^{-1}z_j \) we get
0 \leq t^{k-r} \gamma_t^{(k)}(x, x)
\leq O(1) \int e(z_1)e(z_1-z_2) \ldots e(z_{k-1})(1+|z_1|^N)^{-1}(1+|z_1-z_2|^N)^{-1} \ldots
(1+|z_{k-1}|^N)^{-1} dz_1 \ldots dz_{k-1}.

If 2mk > n, we can make the right side finite by choosing N large enough and ε so small that 2mk - εk > n. Combining this formula with (4.7) and applying Lebesgue’s theorem it follows that
\lim t^{k-r} \int_{\mathbb{R}^n} \gamma_t^{(k)}(x, x) dx = (2\pi)^{-n} \int_{\mathbb{R}^n} dx \int (a_0(x, \xi) + 1)^{-k} d\xi.

Since \gamma_t^{(k)}(x, y) is a continuous and positive kernel, the integral on the left side is the trace of \Gamma_t^{k}, and hence
\lim t^{k-r} \text{tr} \ G_t^{k} \leq (2\pi)^{-n} \int_{\mathbb{R}^n} dx \int (a_0(x, \xi) + 1)^{-k} d\xi.

It follows from this formula and (4.5) that
(4.7) \lim_{t \to \infty} t^{k-r} \text{tr} \ G_t^{k} = (2\pi)^{-n} \int_{\mathbb{R}^n} dx \int (a_0(x, \xi) + 1)^{-k} d\xi
when 2mk > n.

Since \ G_t is self-adjoint and positive, there exists a complete system \varphi_1, \varphi_2, \ldots of eigenfunctions of \ G_t (t fixed) with positive eigenvalues \(\lambda_1 + t\)^{-1} \geq (\lambda_2 + t)^{-1} \geq \ldots. In view of the properties of \ G_t, a necessary and sufficient condition that \varphi \in \mathcal{H}_0 and \ G_t \varphi = (\lambda + t)^{-1} \varphi is that \varphi \in \mathcal{H} and that \(\lambda + t)(\varphi, f) = a_t(\varphi, f)\) for all \ f in \mathcal{H}. Hence \ G_t \varphi = (\lambda + t)^{-1} \varphi implies \ G_s \varphi = (\lambda + s)^{-1} \varphi and conversely if \ t and \ s are large enough. Hence
\ G_t \varphi_j = (\lambda_j + t)^{-1} \varphi_j
for all \ t. Moreover, \(\lambda_j + t)(\varphi_j, f) = a_t(\varphi_j, f)\) means that \(\varphi_j, (\lambda_j - a)f\) = 0 when \ f is in \mathcal{H}(\mathcal{S}), and consequently by Schwartz’s theorem on weak solutions of elliptic differential equations [10] and John’s construction of a fundamental solution, \varphi_j is infinitely differentiable and
a \varphi_j = \lambda_j \varphi_j.

Suppose now for a moment that we have shown that
(4.8) \ g_t^{(k)}(x, y) = \sum (\lambda_j + t)^{-k} \overline{\varphi_j(x)} \varphi_j(y)
when 2mk > n. Then
(4.9) \int_{\mathbb{R}^n} g_t^{(k)}(x, x) dx = \sum (\lambda_j + t)^{-k} = \text{tr} \ G_t^{k}.

Let us consider the integral
\[
\begin{align*}
 w^{(k)}_a(x) &= (2\pi)^{-n} \int (a_0(x, \xi) + 1)^{-k} d\xi.
\end{align*}
\]

Introducing polar coordinates in the integral by the formula \(d\xi = dq^n d\omega\), where \(q^{2m} = a_0(x, \xi)\), we get, since \(d\omega(\xi)\) is homogeneous of order zero,

\[
 w_a(x) = \int_{a_0(x, \xi) < 1} d\xi = \int_0^1 dq^n \int_0^\infty d\omega = \int d\omega,
\]

and consequently

\[
 w^{(k)}_a(x) = w_a(x) (2\pi)^{-n} \int_0^\infty (q^{2m} + 1)^{-k} dq^n =
\]

\[
 = w_a(x) (2\pi)^{-n} \Gamma(v+1) \Gamma(k-v) (\Gamma(k))^{-1}.
\]

Hence, by (4.4) and (4.8)

\[
(4.10) \quad \sum (\lambda_j + t)^{-k} p_j(x)p_j(y) = (2\pi)^{-n} \delta_{xy} w_a(x) \Gamma(v+1) \Gamma(k-v) (\Gamma(k))^{-1} t^{-k} (1 + o(1)).
\]

Putting \(x = y\) and integrating, we obtain by virtue of (4.9) and (4.7),

\[
\sum (\lambda_j + t)^{-k} = (2\pi)^{-n} w_a(S) \Gamma(v+1) \Gamma(k-v) (\Gamma(k))^{-1} t^{-k} (1 + o(1)),
\]

where \(w_a(S) = \int_S w_a(x) dx\).

By application of a Tauberian theorem of Hardy and Littlewood [4] in the formulation of Pleijel [9], we arrive at the desired formula (0.6) of the introduction. Applying the same theorem to (4.4) with \(x = y\) we get

\[
(4.11) \quad \sum_{\lambda_j \leq t} |p_j(x)|^2 = (2\pi)^{-n} w_a(x) t^{v} (1 + o(1)),
\]

and applying it to

\[
\sum (\lambda_j + t)^{-k} p_j(x) + \theta p_j(y) |^2 = (2\pi)^{-n} (w_a(x) + w_a(y)) \Gamma(v+1) \Gamma(k-v) (\Gamma(k))^{-1} t^{-k} (1 + o(1))
\]

(\(|\theta| = 1\)), which when \(x \neq y\) follows from (4.4), we obtain

\[
(4.12) \quad \sum_{\lambda_j \leq t} |p_j(x) + \theta p_j(y)|^2 = (2\pi)^{-n} (w_a(x) + w_a(y)) t^{v} (1 + o(1)) \quad (|\theta| = 1).
\]

The validity of the formulas (4.11), (4.12) and (0.5) prove the desired formula (0.7).

It remains to prove that (4.8) holds. Consider the kernel \(g^{(k)}_t(x, y)\). Locally it is \(O(1)\) in \(S \times S\) if \(2mk > n\) and \(O(1)|x-y|^{2mk-n-\varepsilon}\) if \(2mk \leq n\). Hence, if \(2mk-n-\varepsilon > -\frac{1}{2}n\), that is, if \(2mk > \frac{1}{2}n + \varepsilon\), the integral

\[
\int_V |g^{(k)}_t(x, y)|^2 dy
\]
is finite provided that \( \bar{V} \) is contained in \( S \). We want to prove that also

\[
(4.13) \quad \int_{S} |g_t^{(k)}(x, y)|^2 \, dy < \infty .
\]

Let us put \( C(f, g) = t^k (G_t^k f, \bar{g}) \) and \( b = t^{-k} a_t^k \), and apply the methods of the preceding section. Let \( U \) be an open subset of \( S \) whose closure \( \bar{U} \) is contained in \( S \) and choose another open set \( U_1 \) such that \( \bar{U} \subset U_1 \subset \bar{U}_1 \subset S \). Let \( \eta \in H(U_1) \) be 1 on \( U \) and put with large \( \tau = t^{1/(2m)} \)

\[
g(x, z) = b(\tau, z, D_z (I(\tau, x, z)(1-\eta(z))), \quad x \in U ,
\]

and

\[
r(x) = C(q(x, \cdot), g) , \quad x \in U ,
\]

where \( g \in H(S - \bar{U}_1) \). If \( f \in H(U) \), it follows from the properties of \( C \) and \( I' \) that

\[
\int r(x) f(x) \, dx = C(q(f, \cdot), g) = C(b(I(\tau, f, \cdot)(1-\eta(\cdot)), g)
\]

\[
= C(f, g) - (I(f, \cdot) \eta(\cdot), g) = C(f, g) .
\]

Now the bilinear form \( C \) has a kernel \( c(\tau, x, z) \) so that the last result can be written in the form

\[
\int r(x) f(x) \, dx = \int c(\tau, x, z) f(x) g(z) \, dx \, dz .
\]

Since \( f \) is arbitrary in \( H(U) \), \( r \) is continuous and \( c \) continuous when \( x \neq z \), we get

\[
C(q(x, \cdot), g) = \int c(\tau, x, z) g(z) \, dz
\]

when \( x \in \bar{U} \) and \( g \in H(S - \bar{U}_1) \). This proves that

\[
\int_{S - \bar{U}_1} |c(\tau, x, z)|^2 \, dz \leq |C|^2 |q(x, \cdot)|^2
\]

when \( x \) is in \( U \). Because \( g_t^{(k)}(x, y) = t^{-k} c(\tau, x, y) \), the formula (4.13) follows and we also see that \( g_t^{(k)}(x, \cdot) \) considered as an element of \( \mathcal{S}_0 \) is uniformly continuous in \( x \) on compact subsets of \( S \). Now by Fubini’s theorem and the properties of \( G_t \),

\[
(\lambda_j + t)^{-k} (f, \varphi_j) = (G_t^k f, \varphi_j) = \int \left\{ \int g_t^{(k)}(x, z) f(x) \, dx \right\} \varphi_j(z) \, dz
\]

\[
= \int \left\{ \int g_t^{(k)}(x, z) \varphi_j(z) \, dz \right\} f(x) \, dx
\]

when \( f \in H(S) \). This means that

\[
\int g_t^{(k)}(x, z) \varphi_j(z) \, dz = (\lambda_j + t)^{-k} \varphi_j(x) ,
\]
both sides being continuous in $x$. Hence, by Parseval’s formula,

$$(4.14) \quad \left(g_t^{(k)}(x, \cdot), g_t^{(k)}(y, \cdot)\right) = \sum (\lambda_j + t)^{-2k} \varphi_j(x) \varphi_j(y).$$

By Fubini’s theorem

$$\int f(x)\bar{f}(y) \, dx \, dy \int g_t^{(k)}(x, z) \bar{g}_t^{(k)}(y, z) \, dz$$

$$= \int dz \left\{ \int g_t^{(k)}(x, z)f(x) \, dx \right\} \left\{ \int g_t^{(k)}(y, z)f(y) \, dy \right\} = (G_t^k f, G_t^k f)$$

$$= (G_t^{2k} f, f) = \int g_t^{(2k)}(x, y)f(x)\bar{f}(y) \, dx \, dy$$

when $f$ is in $H(S)$. Hence the left side of (4.14) equals $g_t^{(2k)}(x, y)$, and since $\varepsilon > 0$ is arbitrary, this proves the desired formula (4.8) when $2mk > n$.

**BIBLIOGRAPHY**


UNIVERSITY OF LUND, SWEDEN