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# APPLICATION OF LOCALIZATION TO THE MULTIVARIATE MOMENT PROBLEM II

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## Abstract

The paper is a sequel to the paper [5], Math. Scand. 115 (2014), 269–286, by the same author. A new criterion is presented for a PSD linear map $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ to correspond to a positive Borel measure on $\mathbb{R}^n$. The criterion is stronger than Nussbaum's criterion (Ark. Math. 6 (1965), 171–191) and is similar in nature to Schmüdgen's criterion in Marshall [5] and Schmüdgen, Ark. Math. 29 (1991), 277–284. It is also explained how the criterion allows one to understand the support of the associated measure in terms of the non-negativity of $L$ on a quadratic module of $\mathbb{R}[\underline{x}]$. This latter result extends a result of Lasserre, Trans. Amer. Math. Soc. 365 (2013), 2489–2504. The techniques employed are the same localization techniques employed already in Marshall (Cand. Math. Bull. 46 (2003), 400–418, and [5]), specifically one works in the localization of $\mathbb{R}[\underline{x}]$ at $p = \prod_{i=1}^{n}(1 + x_i^2)$ or $p' = \prod_{i=1}^{n-1}(1 + x_i^2)$.


This paper is a sequel to the earlier paper [5]. We present a couple of interesting and illuminating results which were inadvertently overlooked when [5] was written; see Theorems 1 and 5 below. Theorem 1 extends an old result of Nussbaum in [6]. See Theorem 3 below for a statement of Nussbaum's result. The density condition (1) appearing in Theorem 1 is weaker than the Carleman condition (2) appearing in Nussbaum's result. Theorem 5 shows how condition (1) allows one to read off information about the support of the measure from the non-negativity of the linear functional on a quadratic module. This illustrates how natural condition (1) is. Theorem 5 extends a result of Lasserre in [3].

We recall some terminology and notation from [4] and [5]. For an $\mathbb{R}$-algebra $A$ (commutative with 1), a *quadratic module* of $A$ is a subset $M$ of $A$ such that $1 \in M$, $M + M \subseteq M$ and $f^2 M \subseteq M$, for all $f \in A$. We let $\sum A^2$ denote the set of all (finite) sums of squares of $A$. Then $\sum A^2$ is the unique smallest quadratic module of $A$. A linear map $L\colon A \to \mathbb{R}$ is said to be PSD (positive semidefinite) if $L(f^2) \geq 0$ for all $f \in A$, equivalently, if $L\big(\sum A^2\big) \subseteq [0, \infty)$. Define $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \dots, x_n]$, $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$. If $\mu$ is a positive Borel measure on $\mathbb{R}^n$ having finite moments, i.e., $\int \underline{x}^k d\mu$ is well-defined and finite for all monomials $\underline{x}^k := x_1^{k_1} \dots x_n^{k_n}$, $k_j \geq 0$, $j = 1, \dots, n$, the PSD linear map

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$L_\mu\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is defined by $L_\mu(f) = \int f\, d\mu$. If $\nu$ is another positive Borel measure on $\mathbb{R}^n$ having finite moments then we write $\mu \sim \nu$ is indicate that $\mu$ and $\nu$ have the same moments, i.e., $L_\mu = L_\nu$. We say $\mu$ is *determinate* if $\mu \sim \nu \Rightarrow \mu = \nu$.

THEOREM 1. *Suppose $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD and, for $j = 1, \dots, n-1$,*

(1) $\exists$ *a sequence $\{q_{jk}\}_{k=1}^\infty$ in $\mathbb{C}[\underline{x}]$ such that*

$$\lim_{k\to\infty} L\big(|1-(1+x_j^2)q_{jk}\overline{q_{jk}}|^2\big) = 0.$$

*Then there exists a positive Borel measure $\mu$ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition* (1) *holds also for $j = n$ then the measure is determinate.*

PROOF. Extend $L$ to $\mathbb{C}[\underline{x}]$ in the obvious way, i.e., $L(f_1 + if_2) := L(f_1) + iL(f_2)$. Define $\langle f, g\rangle := L(f\overline{g})$, $\|f\| := \sqrt{\langle f, f\rangle}$. According to [5, Corollary 4.8] to prove the existence assertion it suffices to show that $\forall\, g \in \mathbb{C}[\underline{x}]$ and $\forall\, j = 1, \dots, n-1$,

$$\lim_{k\to\infty} L(g(1-(1+x_j^2)q_{jk}\overline{q_{jk}})) = 0.$$

This is immediate from condition (1), using the Cauchy-Schwartz inequality. According to [5, Corollary 2.7], to show uniqueness it suffices to show $\forall\, j = 1, \dots, n$ $\exists$ a sequence $\{p_{jk}\}_{k=1}^\infty$ in $\mathbb{C}[\underline{x}]$ such that

$$\lim_{k\to\infty} L\big(|1-(x_j - i)p_{jk}|^2\big) = 0.$$

Uniqueness follows from this criterion, taking $p_{jk} := (x_j + i)q_{jk}\overline{q_{jk}}$.

We remark that [5, Theorem 4.9] is a consequence of Theorem 1. This is immediate from the following:

LEMMA 2. *Suppose $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD. Suppose $\{q_{jk}\}_{k=1}^\infty$ is a sequence of polynomials in $\mathbb{C}[\underline{x}]$. Then*

$$\lim_{k\to\infty} L\big(|1-(x_j - i)q_{jk}|^4\big) = 0 \implies \lim_{k\to\infty} L\big(|1-(1+x_j^2)q_{jk}\overline{q_{jk}}|^2\big) = 0.$$

PROOF. Let $Q_k := 1 - (x_j - i)q_{jk}$. Thus

$$1-(1+x_j^2)q_{jk}\overline{q_{jk}} = 1-(1-Q_k)(1-\overline{Q}_k) = Q_k + \overline{Q}_k - Q_k\overline{Q_k}.$$

We are assuming $\|Q_k\overline{Q_k}\| \to 0$ as $k \to \infty$ and we want to show $\|Q_k + \overline{Q_k} - Q_k\overline{Q_k}\| \to 0$ as $k \to \infty$. Applying the Cauchy-Schwartz inequality and the triangle inequality we obtain $\|Q_k\|^2 = \|\overline{Q_k}\|^2 = \langle Q_k\overline{Q_k}, 1\rangle \leq \|Q_k\overline{Q_k}\| \cdot \|1\|$ and

$$\begin{aligned}\|Q_k + \overline{Q_k} - Q_k\overline{Q_k}\| &\leq \|Q_k\| + \|\overline{Q_k}\| + \|Q_k\overline{Q_k}\| \\ &\leq 2\sqrt{\|Q_k\overline{Q_k}\| \cdot \|1\|} + \|Q_k\overline{Q_k}\|.\end{aligned}$$

At this point the result is clear.

The following result of Nussbaum [6, Theorem 4.11] can also be seen as a consequence of Theorem 1.

THEOREM 3 (Nussbaum). *Suppose $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD and, for $j = 1, \dots, n-1$, the Carleman condition*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{[2k][b]L(x_j^{2k})}} = \infty \tag{2}$$

*holds. Then there exists a positive Borel measure $\mu$ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition* (2) *holds also for $j = n$ then the measure is determinate.*

PROOF. Argue as in [5, Theorem 4.10]. Let $\mu_j$ be the positive Borel measure on $\mathbb{R}$ such that $L_{\mu_j} = L|_{\mathbb{R}[x_j]}$. According to [1, Théorème 3], the Carleman condition (2) implies that $\mathbb{C}[x_j]$ is dense in the Lebesgue space $\mathscr{L}^s(\mu_j)$ for all $s \in [1, \infty)$. Fix $s > 4$. Thus $\exists\, q_{jk} \in \mathbb{C}[x_j]$ such that $\lim_{k\to\infty} \|q_{jk} - 1/(x_j - i)\|_{s,\mu_i} = 0$. An easy application of Hölder's inequality (taking $p = s/4$, $q = s/(s-4)$) shows that

$$\begin{aligned}L(|1 - (x_j - i)q_{jk}|^4) &= \int \left| q_{jk} - \frac{1}{x_j - i}\right|^4 |x_j - i|^4 \, d\mu_j \\ &\leq \left[ \left\| q_{jk} - \frac{1}{x_j - i}\right\|_{s,\mu_j} \cdot \|x - i\|_{4s/(s-4),\mu_j} \right]^4\end{aligned}$$

so $\lim_{k\to\infty} L\big(|1 - (x_j - i)q_{jk}|^4\big) = 0$. The result follows now, by Lemma 2 and Theorem 1.

The reader should compare Theorems 1 and 3 with the following result of Schmüdgen [5, Theorem 4.11] [7, Proposition 1], which, according to Fuglede [2, p. 62], is an unpublished result of J. P. R. Christensen, 1981.

THEOREM 4 (Schmüdgen). *Suppose $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD. Fix a positive Borel measure $\mu_j$ on $\mathbb{R}$ such that $L|_{\mathbb{R}[x_j]} = L_{\mu_j}$ and suppose for $j = 1, \dots, n-1$ that $\mathbb{C}[x_j]$ is dense in $\mathscr{L}^4(\mu_j)$, i.e.,*

$$\exists \text{ a sequence } \{q_{jk}\}_{k=1}^{\infty} \text{ in } \mathbb{C}[x_j] \text{ such that } \lim_{k\to\infty} \left\| q_{jk} - \frac{1}{x_j - i}\right\|_{4,\mu_j} = 0. \tag{3}$$

*Then there exists a positive Borel measure $\mu$ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition* (3) *holds also for $j = n$ then the measure is determinate.*

By considering products of measures of the sort considered by Sodin in [8], one sees that Theorem 1 and Theorem 4 are both strictly stronger than Nussbaum's result. But it is not clear, to the author at least, how Theorems 1 and 4 are related. In particular, it is not clear that either result implies the other.

We turn now to the problem of describing the support of $\mu$. By definition, the support of $\mu$ is the smallest closed set $K$ of $\mathbb{R}^n$ satisfying $\mu(\mathbb{R}^n \backslash K) = 0$. We recall additional notation from [4] and [5]. If $M$ is a quadratic module of an $\mathbb{R}$-algebra $A$, define

$$X_M := \{\alpha : A \to \mathbb{R} \mid \alpha \text{ is an } \mathbb{R}\text{-algebra homomorphism}, \alpha(M) \subseteq [0, \infty)\}.$$

If $M = \sum A^2 + I$, where $I$ is an ideal of $A$, the condition $\alpha(M) \subseteq [0, \infty)$ is equivalent to the condition $\alpha(I) = \{0\}$. Let $\mathbb{R}[\underline{x}]_p$ denote the localization of $\mathbb{R}[\underline{x}]$ at $p$, where $p := \prod_{j=1}^n (1 + x_j^2)$. If $A$ is $\mathbb{R}[\underline{x}]$ or $\mathbb{R}[\underline{x}]_p$ then algebra homomorphisms $\alpha \colon A \to \mathbb{R}$ are identified with points of $\mathbb{R}^n$ via the map $\alpha \mapsto (\alpha(x_1), \dots, \alpha(x_n))$ and $X_M$ is identified with the set $\{\underline{a} \in \mathbb{R}^n \mid g(\underline{a}) \geq 0 \ \forall\, g \in M\}$.

THEOREM 5. *Suppose $L \colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ is a PSD linear map satisfying* (1) *for $j = 1, \dots, n$ and $g \in \mathbb{R}[\underline{x}]$ is such that $L(gh^2) \geq 0 \ \forall\, h \in \mathbb{R}[\underline{x}]$. Then the support of the associated positive Borel measure $\mu$ is contained in the set $\{\underline{a} \in \mathbb{R}^n \mid g(\underline{a}) \geq 0\}$.*

See [3, Theorem 2.2] for an earlier version of this result.

PROOF. Denote by $L \colon \mathbb{R}[\underline{x}]_p \to \mathbb{R}$ the PSD linear extension of $L$ defined by $L(f) := \int f \, d\mu \ \forall\, f \in \mathbb{R}[\underline{x}]_p$.

We claim that $L(gh\overline{h}) \geq 0 \ \forall\, h \in \mathbb{C}[\underline{x}]_p$ (so, in particular, $L(gh^2) \geq 0 \ \forall\, h \in \mathbb{R}[\underline{x}]_p$). The proof is by induction of the number of factors of the form $x_j \pm i$, $j = 1, \dots, n$, appearing in the denominator of $h$. Suppose $x_j \pm i$ appears in the denominator of $h$. Note that $(x_j \pm i)hq_{jk}$ has fewer factors $x_j \pm i$ appearing in the denominator, so, by induction, $L\big(g(1 + x_j^2)h\overline{h}q_{jk}\overline{q_{jk}}\big) \geq 0$. Applying the Cauchy-Schwartz inequality, we see that $L\big(gh\overline{h}(1 - (1 + x_j^2)q_{jk}\overline{q_{jk}})\big) \to 0$ as $k \to \infty$. It follows that $L\big(g(1 + x_j^2)h\overline{h}q_{jk}\overline{q_{jk}}\big) \to L(gh\overline{h})$ as $k \to \infty$, so $L(gh\overline{h}) \geq 0$. This proves the claim.

Denote by $Q$ the quadratic module of $\mathbb{R}[\underline{x}]_p$ generated by $g$, i.e., $Q := \sum \mathbb{R}[\underline{x}]_p^2 + \sum \mathbb{R}[\underline{x}]_p^2 g$. It follows from the claim together with the fact that $L$ is PSD on $\mathbb{R}[\underline{x}]_p$ that $L(Q) \subseteq [0, \infty)$. By [4, Corollary 3.4] there exists a positive Borel measure $\nu$ on $X_Q = \{\underline{a} \in \mathbb{R}^n \mid g(\underline{a}) \geq 0\}$ such that $L(f) = \int f \, d\nu \ \forall\, f \in \mathbb{R}[\underline{x}]_p$. Uniqueness of $\mu$ implies $\mu = \nu$.

COROLLARY 6. *If $L$ satisfies condition* (1) *for* $j = 1, \dots, n$ *and* $L(M) \subseteq [0, \infty)$ *for some quadratic module $M$ of* $\mathbb{R}[\underline{x}]$*, then the support of the associated positive Borel measure $\mu$ is contained in the set* $X_M = \{\underline{a} \in \mathbb{R}^n \mid g(\underline{a}) \geq 0 \; \forall \, g \in M\}$.

REMARK 7. (1) The quadratic module $M$ is not required to be finitely generated, although this seems to be the most interesting case.

(2) For a quadratic module of the form $M = \sum \mathbb{R}[\underline{x}]^2 + I$, with $I$ an ideal of $\mathbb{R}[\underline{x}]$, one can weaken the hypothesis. It is no longer necessary to assume that $L$ satisfies condition (1) for $j = 1, \dots, n$ but only that $L = L_\mu$. This is more or less clear. By the Cauchy-Schwartz inequality, for $g \in \mathbb{R}[\underline{x}]$,

$$L(gh) = 0 \; \forall \, h \in \mathbb{R}[\underline{x}] \iff L(g^2) = 0 \iff L(gh) = 0 \; \forall \, h \in \mathbb{R}[\underline{x}]_p.$$

Also, in this case, $X_M = Z(I) = \{\underline{a} \in \mathbb{R}^n \mid g(\underline{a}) = 0 \; \forall \, g \in I\}$.

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