WEAK COMPACTNESS IN THE DUAL SPACE OF A JB*-TRIPLE IS COMMUTATIVELY DETERMINED

FRANCISCO J. FERNÁNDEZ-POLO and ANTONIO M. PERALTA*

Abstract
We prove the following criterium of weak compactness in the dual of a JB*-triple: a bounded set \( K \) in the dual of a JB*-triple \( E \) is not relatively weakly compact if and only if there exist a sequence of pairwise orthogonal elements \((a_n)\) in the closed unit ball of \( E \), a sequence \((\varphi_n)\) in \( K \), and \( \vartheta > 0 \) satisfying that \( |\varphi_n(a_n)| > \vartheta \) for all \( n \in \mathbb{N} \). This solves a question stimulated by the main result in [11] and posed in [9].

1. Introduction and Preliminaries
Relatively weakly compact subsets in the dual of a C*-algebra have been intensively studied during the last fifty years. The first precedent appears in a paper by A. Grothendieck in 1953 (see [15]). This forerunner establishes the following characterization of weak compactness in the dual of a \( C(\Omega) \)-space: a bounded subset \( K \subseteq C(\Omega)^* \) is not relatively weakly compact if and only if there exists a sequence \((O_n)\) of pairwise disjoint open subsets of \( \Omega \) such that \( \lim_{n \to \infty} \sup \{ |\mu(O_n)| : \mu \in K \} \neq 0 \). Urysohn’s lemma allows us to replace the \( O_n \)’s by norm-one positive continuous functions on \( \Omega \) with mutually disjoint supports.

When \( K \) is a bounded set in the predual of a von Neumann algebra \( M \), M. Takesaki [26] and C. Akemann [1] (see also [27, Theorem III.5.4]) proved that \( K \) is not relatively weakly compact if and only if there exists a sequence \((p_n)\) of pairwise orthogonal projections in \( M \) such that \( \lim_{n \to \infty} \sup \{ |\phi(p_n)| : \phi \in K \} \neq 0 \). That is, weak compactness in \( M^* \) is determined by the abelian subalgebras of \( M \). Consequently, relatively weakly compact subsets in the dual of a C*-algebra \( A \) are commutatively determined by the abelian subalgebras of \( A^{**} \).

In [24] H. Pfitzner showed that weak compactness in the dual of a C*-algebra \( A \) is in fact determined by the abelian subalgebras of \( A \). Concretely, a bounded set \( K \subseteq A^* \) fails to be relatively weakly compact if and only if there

* Authors partially supported by M.I.C. project no. MTM2008-02186, and Junta de Andalucía grants FQM0199 and FQM1215.

Received June 9, 2008; in final form October 15, 2008.
exist a positive $\theta$, a sequence $(a_n)$ of pairwise orthogonal positive elements in the closed unit ball of $A$ and a sequence $(\varphi_n)$ in $K$ satisfying $|\varphi_n(a_n)| > \theta$, for every $n \in \mathbb{N}$ (compare [12] for a new and shorter proof).

C*-algebras belong to a more general class of complex Banach spaces in which the geometric, holomorphic, and algebraic structure mutually interplay. We are referring to the class of JB*-triples. We recall (see [21]) that a JB*-triple is a complex Banach space $E$ equipped with a continuous triple product $\{ \cdot, \cdot, \cdot \} : E \times E \times E \to E$, which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies:

(i) (Jordan Identity) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b)$ is the operator on $E$ given by $L(a, b)x = \{ a, b, x \}$;

(ii) $L(a, a)$ is a hermitian operator with non-negative spectrum;

(iii) $\| L(a, a) \| = \| a \|^2$.

Every C*-algebra is a JB*-triple with respect to the product $\{ x, y, z \} = \frac{1}{2}(xy^*z + zy^*x)$, and every JB*-algebra is a JB*-triple under the triple product $\{ x, y, z \} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [8]). Further, the triple product of every JBW*-triple is separately weak*-continuous [3].

The above quoted results of Takesaki and Akemann were extended in [23] to characterize relatively weakly compact subsets in the predual of a JBW*-triple.

A JC*-triple is a norm-closed subspace of a C*-algebra which is closed under the ternary product $\{ x, y, z \} = \frac{1}{2}(xy^*z + zy^*x)$. JC*-triples form an intermediate class of complex Banach spaces between C*-algebras and JB*-triples. A criterium for weak compactness in the dual of a JC*-triple, which is also a generalization of Pfitzner’s result, was established in [11]. This criterium assures that a bounded subset in the dual space of a JC*-triple $E$ is relatively weakly compact if and only if its restriction to any abelian maximal subtriple $C$ of $E$ is relatively weakly compact in the dual of $C$. However, as pointed out by C. M. Edwards in [9], “whether the results hold for general JB*-triples remains an open question”. The main result of this paper gives a positive answer to this question for general JB*-triples (see Theorem 2.3). The solution presented in this paper is itself a novelty which simplifies the results in [11] with a new and shorter orthogonalization process based on Bergmann operators.

Reference [6] is a basic forerunner of the problem studied in this paper. Briefly speaking, we could say [6] contains a partial answer for our problem in terms of Pelczynski’s Property (V). We recall that a series $\sum_{n \geq 1} z_n$ in a Banach space $X$ is called weakly unconditionally convergent (w.u.c. for short)
if for each \( \varphi \in X^* \) we have \( \sum_{n=1}^{\infty} |\varphi(z_n)| < \infty \), equivalently, there exists \( C > 0 \) such that for any finite subset \( \mathcal{F} \subseteq \mathbb{N} \) and \( |\varepsilon_k| = 1 \) in \( C \) we have \( \| \sum_{k \in \mathcal{F}} \varepsilon_k z_k \| \leq C \), (see, for example, [7, Theorem 6 in Chapter 5]). It is clear that every bounded linear operator between Banach spaces preserves w.u.c. series. A Banach space \( X \) has property \((V)\) if for any (bounded) non relatively weakly compact set \( K \subseteq X^* \) there exists a w.u.c. series \( \sum_n x_n \) in \( X \) such that \( \sup_{\varphi \in X^*} |\varphi(x_n)| \) does not converge to zero. It is established in [6] that every JB*-triple satisfies property \((V)\). We shall see later that every bounded sequence of mutually orthogonal elements in a JB*-triple defines a w.u.c. series, however the reciprocal statement need not hold in general. We shall establish a new orthogonalization method to construct sequences of mutually orthogonal elements from w.u.c. series.

1.1. Preliminaries

Let \( X \) and \( Y \) be two Banach spaces, throughout the paper, the symbol \( L(X, Y) \) will stand for the space of all bounded linear operators from \( X \) to \( Y \). We shall write \( L(X) \) for the space \( L(X, X) \).

A JB*-triple \( E \) is said to be abelian if\( \{\{x, y, z]\}, u, v\} = \{\{x, y\}, \{z, u, v\}\} \), for all \( x, y, z, u, v \in E \). The JB*-subtriple generated by a single element is always abelian.

Let \( x \) be an element in a JB*-triple \( E \). Throughout the paper the symbol \( E_x \) will denote the norm-closed subtriple of \( E \) generated by \( x \). It is known that \( E_x \) is JB*-triple isomorphic to the \( C^* \)-algebra \( C_0(L) \) of all complex-valued continuous functions on \( L \) vanishing at 0, where \( L \) is a locally compact subset of \( (0, \|x\|] \) satisfying that \( L \cup \{0\} \) is compact. Further, there exists a JB*-triple isomorphism \( \Psi : E_x \to C_0(L) \) which satisfies \( \Psi(x)(t) = t \), for all \( t \) in \( L \) (compare [20, 4.8] and [21, 1.15]). In particular, given a natural \( n \), the symbol \( x^{\frac{-1}{n+1}} \) makes sense as an element of \( E_x \cong C_0(L) \).

An element \( u \) in a JB*-triple \( E \) is said to be a tripotent if \( u = \{u, u, u\} \). Given a tripotent \( u \in E \), the mappings \( P_i(u) : E \to E_i, (i = 0, 1, 2) \), defined by

\[
P_2(u) = L(u, u)(2L(u, u) - \text{id}_E),
\]
\[
P_1(u) = 4L(u, u)(\text{id}_E - L(u, u)), \quad \text{and}
\]
\[
P_0(u) = (\text{id}_E - L(u, u))(\text{id}_E - 2L(u, u)),
\]

are contractive linear operators. For each \( j = 0, 1, 2 \), \( P_j(u) \) is the projection onto the eigenspace \( E_j(u) \) of \( L(u, u) \) corresponding to the eigenvalue \( \frac{j}{2} \) and

\[
E = E_2(u) \oplus E_1(u) \oplus E_0(u)
\]

is the Peirce decomposition of \( E \) relative to \( u \). Furthermore, the following
Peirce rules are satisfied,

\begin{align}
\{E_2(u), E_0(u), E \} &= \{E_0(u), E_2(u), E \} = 0, \\
\{E_i(u), E_j(u), E_k(u)\} &\subseteq E_{i-j+k}(u),
\end{align}

where $E_{i-j+k}(u) = 0$ whenever $i-j+k \notin \{0, 1, 2\}$ (compare [13]).

When $W$ is a JBW*-triple, the JBW*-subtriple generated by a norm-one element $x \in W$ coincides with the weak*-closure, $\overline{W_x}^w$, of $W_x$. By [18, Lemma 3.11] there exists a JBW*-triple isomorphism, $\Psi$, between $W_x$ and a commutative $W^*$-algebra $C$. We shall write $r(x) = \Psi^{-1}(1)$, where 1 denotes the unit element in $C$. It is clear that $r(x)$, commonly termed the range tripotent of $x$, is a tripotent in $W$. Moreover, $r(x)$ coincides with the weak*-limit of the sequence $x^{\frac{1}{n}}$, $(n \in \mathbb{N})$. It is also known that the JBW*-algebra $E_2^*(r(x))$ contains $x$ as a positive element (compare [10]).

Given a JBW*-triple $W$, a norm-one element $\varphi$ in $W^*$ and a norm-one element $z$ in $W$ with $\varphi(z) = 1$, it follows from [2, Proposition 1.2] that the assignment

$$(x, y) \mapsto \varphi \{x, y, z\}$$

defines a positive sesquilinear form on $W$. Further, for every norm-one element $w$ in $W$ satisfying $\varphi(w) = 1$, we have $\varphi \{x, y, z\} = \varphi \{x, y, w\}$, for all $x, y \in W$. The mapping $x \mapsto \|x\|_\varphi := (\varphi \{x, x, z\})^{\frac{1}{2}}$, defines a prehilbertian seminorm on $W$. The Strong*-topology (noted by $S^*(W, W^*)$) is the topology on $W$ generated by the family \{ $\|\cdot\|_\varphi : \varphi \in W^*, \|\varphi\| = 1$\}. This topology was introduced by T. J. Barton and Y. Friedman in [2].

When $\varphi$ is an element in the dual of a JB*-triple $E$, the prehilbertian seminorm $\|\cdot\|_\varphi$ is defined on $E^{**}$ (and hence on $E$) by the assignment

$$x \mapsto \|x\|_\varphi := (\varphi \{x, x, z\})^{\frac{1}{2}},$$

where $z$ is a norm-one element in $E^{**}$ with $\varphi(z) = \|\varphi\|$. The inequality

$$\|\{x, y, z\}\| \leq \|x\| \|y\| \|z\|$$

holds for every $x, y$ and $z$ in a JB*-triple $E$ (compare [14, Corollary 3]). Consequently,

$$\|x\|_\varphi \leq \|\varphi\|^{\frac{1}{2}} \|x\|,$$

for all $\varphi \in E^*$ and $x \in E$.

For each element $a$ in a JB*-triple $E$, the conjugate linear mapping $Q(a)$ from $E$ to itself is defined, for each element $b$ in $E$, by $Q(a)(b) := \{a, b, a\}$. Let $x, y$ be two elements in $E$. The Bergmann operator $B(x, y) : E \rightarrow E$
is defined by $B(x,y)(z) = z - 2L(x,y)(z) + Q(x)Q(y)(z)$, for all $z$ in $E$ (compare [22] or [28, page 305]). In the particular case of $u$ being a tripotent, it is known that $P_0(u) = B(u,u)$.

Let $x$ be a symmetric element in a unital JB*-algebra $A$. The operator $U_x : A \to A$ is defined by $U_x(y) := 2(y \circ x) - x^2 \circ y$, for all $y$ in $A$. When $A$ is regarded as a JB*-triple, we have $U_x(y) = Q(x)(y^*)$, $\forall y \in A$. Since by [16, Lemma 2.4.21] $U_x^2 = U_x$, we deduce that $Q(x)^2(y) = U_x^2(y) = U_x(y) = Q(x^2)(y^*)$, $\forall y \in A$.

We also have $2L(x,x)(y) = 2(x^2 \circ y + (y \circ x) \circ x - (y \circ x) \circ x) = 2x^2 \circ y$, for all $y \in A$. Therefore, for each $y \in A$ we have

$$B(x,x)(y) = y - 2L(x,x)(y) + Q(x^2)(y) = Q(1 - x^2)(y^*),$$

which implies that $\|B(x,x)\| \leq 1$, whenever $x$ belongs to the closed unit ball of $A$.

A tripotent $u$, in a JB*-triple $E$, is said to be bounded if there exists a norm-one element $x \in E$ such that $L(u,u)x = u$. The element $x$ is a bound of $u$ and in this case we write $u \leq x$. We shall write $y \leq u$ whenever $y$ is a positive element in the JB*-algebra $E_2(u)$ (compare [11, pages 79–80]).

**Lemma 1.1.** Let $x$ be a symmetric element in the closed unit ball of a JB*-algebra $A$. Then $B(x,x)$ is a contractive operator. Moreover, if $p$ is a projection in $A$ with $p \leq x$, then $B(x,x)(y)$ belongs to $A_0(p)$, for every $y$ in $A$.

**Proof.** We may assume that $A$ is unital. The comments preceding this lemma guarantee that $\|B(x,x)\| \leq 1$ and $B(x,x)y = Q(1 - x^2)(y^*)$, $(y \in A)$. Since $p \leq x^2 \leq 1$, we have $0 \leq 1 - x^2 \leq 1 - p$, and hence $1 - x^2$ belongs to $A_0(p)$. Finally, it follows, by Peirce rules, that $B(x,x)y \in A_0(p)$.

Lemma 1.1 above can be now extended to JB*-triples.

**Lemma 1.2.** Let $E$ be a JB*-triple, $e$ a tripotent in $E$, and $x$ a norm-one element in $E$ with $e \leq x$. Then $B(x,x)$ is a contractive operator and $B(x,x)(y)$ belongs to $E_0(e)$, for every $y$ in $E$.

**Proof.** By [14, Corollary 1] we may suppose that $E$ embeds as a subtriple into a JBW*-algebra, $A$, of the form $L(H) \oplus \ell^\infty N$, where $H$ is a complex Hilbert space and $N$ is an $\ell^\infty$-sum of finite-dimensional simple JB*-algebras. We may then assume that $e \leq x \leq r(x)$
in the JBW*-algebra $A$, where $r(x)$ is the range tripotent of $x$ in $A$. From [4, Lemma 2.3] and [22, Corollary 5.12] there exists a weak*-continuous isometric triple embedding $T$ from $A$ into $A$, such that $T(r(x))$ (and hence $T(e)$) is a projection in $A$. It is easy to check that $0 \leq T(e) \leq T(x) \leq T(r(x))$. By Lemma 1.1, we have $T(B(x, x)(y)) = B(T(x), T(x))(T(y)) \in A_0(T(e))$, for every $T(y) \in T(E) \subseteq A$. Therefore, $B(x, x)(y) \in A_0(e) \cap E = E_0(e)$, for all $y \in E$.

Another central notion in the paper is the concept of orthogonality. Two elements $a, b$ in a JB*-triple, $E$, are said to be orthogonal (written $a \perp b$) if $L(a, b) = 0$. Lemma 1 in [5] shows that $a \perp b$ if and only if one of the following statements holds:

$$\{a, a, b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b); \quad E_2^{**}(r(a)) \perp E_2^{**}(r(b));$$

$$r(a) \in E_0^{**}(r(b)); \quad a \in E_0^{**}(r(b)); \quad b \in E_0^{**}(r(a)); \quad E_a \perp E_b.$$ 

The Peirce rule (1) shows that for each tripotent $u$ in a JB*-triple $E$, $E_0(u) \perp E_2(u)$. The Jordan identity and the above reformulations assure that

$$(3) \quad a \perp \{x, y, z\}, \quad \text{whenever} \quad a \perp x, y, z.$$ 

Let $A$ be a $C^*$-algebra. Two elements $a, b \in A$ are said to be orthogonal for the $C^*$-algebra product if $ab^* = b^*a = 0$. However, $A$ also enjoys a structure of JB*-triple. We have, a priori, two notions of orthogonality in $A$. It can be checked, from the above reformulations, that two elements $a, b$ in $A$ are orthogonal for the $C^*$-algebra product if and only if they are orthogonal when $A$ is considered as a JB*-triple.

For every tripotent $e$ in a JB*-triple $E$, the formula

$$\|P_2(e)(x) + P_0(e)(x)\| = \max\{\|P_2(e)(x)\|, \|P_0(e)(x)\|\},$$

holds for every $x$ in $E$ (compare [13, Lemma 1.3]). In particular, if $\{x_1, \ldots, x_m\}$ is a set of mutually orthogonal elements in a JB*-triple $E$, it follows from the above equivalent reformulations of orthogonality and the previous formula, that the JB*-subtriple generated by the set $\{x_1, \ldots, x_m\}$ coincides with the $\ell_\infty$-sum $\bigoplus_{k=1}^\infty E_{x_k}$ and hence it is JB*-triple isomorphic to an abelian $C^*$-algebra.

We deduce from the above paragraph that every bounded sequence of pairwise orthogonal elements in a JB*-triple defines a w.u.c. series.
2. Main result

The aim of this section is to prove that weak compactness in the dual of a JB*-triple is commutatively determined. Bergmann operators, wisely used, turn to be a powerful tool in orthogonalization processes. More concretely, we shall make use of appropriated Bergmann operators to orthogonalize weakly unconditional convergent series in JB*-triples.

**Lemma 2.1.** Let $E$ be a JB*-triple, $v$ a tripotent in $E$, and $\varphi$ an element in the closed unit ball of $E^*$. Then for each $y \in E_2(v)$ with $\|y\| \leq 1$ we have

$$|\varphi(x - B(y, y)(x))| < 21 \|x\| \|v\| \varphi,$$

for every $x \in E$.

**Proof.** By Peirce rules we have $L(y, y)(x) \in E_2(v) \oplus E_1(v)$ and $Q(y)^2(x) \in E_2(v)$. Since $x - B(y, y)(x) = 2L(y, y)(x) - Q(y)^2(x)$, the desired statement follows from [11, Lemma 3.2].

We shall also need the following strengthening version of [11, Lemma 3.4].

**Lemma 2.2.** Let $E$ be a JB*-triple, $\theta > 0$, $\delta_n > 0$ ($n \in \mathbb{N}$), and let $\{\varphi_1\} \cup \{\varphi_n\}_{n \geq 2}$ be a sequence of functionals in the closed unit ball of $E^*$. Given an element $x$ in the closed unit ball of $E$, satisfying $|\varphi_1(x)| > \theta$ and $\|x\| \varphi_n < \delta_n$, $n \geq 2$, there exist two elements $a, y$ in the unit ball of $E_x$, and two tripotents $u, v$ in $(E_x)^{**}$ such that $a \leq u \leq y \leq v$, $|\varphi_1(a)| > \frac{3}{4} \theta$, and $\|v\| \varphi_n < \frac{8}{\theta} \delta_n$, $n \geq 2$.

**Proof.** We have already commented that $E_x$ is JB*-triple isomorphic to the C*-algebra $C_0(L)$, where $L$ is a locally compact subset of $(0, \|x\|)$ satisfying that $L \cup \{0\}$ is compact. Moreover, there exists a JB*-triple isomorphism $\Psi : E_x \to C_0(L)$ satisfying $\Psi(x)(t) = t$, for all $t$ in $L$. By slight abuse of notation, $E_x$ and $C_0(L)$ will be identified.

Let $a, y \in C_0(L)$ be the functions defined by

$$a(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\theta}{4} \\ 2t - \frac{\theta}{2}, & \text{if } \frac{\theta}{4} \leq t \leq \frac{\theta}{2} \\ t, & \text{if } \frac{\theta}{2} \leq t \leq \|x\| \end{cases}$$

$$y(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\theta}{8} \\ \frac{8}{\theta} (t - \frac{\theta}{8}), & \text{if } \frac{\theta}{8} \leq t \leq \frac{\theta}{4} \\ 1, & \text{if } \frac{\theta}{4} \leq t \leq \|x\|. \end{cases}$$
Since \(\|x - a\| < \frac{\theta}{4}\) and \(|\varphi_1(x)| > \theta\) it follows that \(|\varphi_1(a)| > \frac{3}{4}\theta\).

The element \(x\) decomposes as the sum of two orthogonal elements \(x = x\chi_{[\frac{\theta}{4}, \|x\|]} + x\chi_{[0, \frac{\theta}{4}]}\) (in \((E_x)^*\)). Since \(\|\cdot\|_{\varphi_n}\) is additive when applied to the sum of orthogonal elements, we get \(\|x\chi_{[\frac{\theta}{4}, \|x\|]}\|_{\varphi_n} < \delta_n\). We define \(a = x\chi_{[\frac{\theta}{4}, \|x\|]}\), \(v = x\chi_{[\frac{\theta}{4}, \|x\|]}\) (in \((E_x)^*\)), which clearly satisfy that \(a \leq u \leq y \leq v\).

Since \(\|\cdot\|_{\varphi}\) is an order-preserving map on the set of positive elements in \((E_x)^*\) ([11, Lemma 3.3]), we have that \(\|v\|\varphi_n \leq \|\frac{8}{\theta}x\chi_{[\frac{\theta}{4}, \|x\|]}\|_{\varphi_n} < \frac{8}{\theta}\delta_n\) \((n \geq 2)\), which finishes the proof.

Our main result can be stated now.

**Theorem 2.3.** Let \(E\) be a JB*-triple and \(K\) be a bounded subset in \(E^*\). The following are equivalent:

a) \(K\) is not relatively weakly compact.

b) There exist a sequence of pairwise orthogonal elements \((a_n)\) in the closed unit ball of \(E\), a sequence \((\varphi_n)\) in \(K\), and \(\vartheta > 0\) satisfying that \(|\varphi_n(a_n)| > \vartheta\) for all \(n \in \mathbb{N}\).

b′) There exists a subtriple \(C\) of \(E\) isometric to an abelian C*-algebra such that the restriction of \(K\) to it is not relatively weakly compact.

**Proof.** a) \(\Rightarrow\) b). Since JB*-triples have Pelczynski’s Property (V) (compare [6]) there exist \(\theta > 0\), \((\varphi_n) \subset K\) and a w.u.c. series \(\sum_{n \geq 1} z_n\) in \(E\) with \(\|z_n\| \leq 1\), such that \(|\varphi_n(z_n)| > \theta\), \(\forall n \in \mathbb{N}\). We may assume that \(K\) is contained in the closed unit ball of \(E^*\).

Let us fix a decreasing sequence \((\delta_n)\) of positive numbers satisfying \(\frac{336}{\theta} \sum_{n=1}^{\infty} \delta_n < \frac{\theta}{2}\). We shall construct, inductively, a sequence \((a_n)\) of mutually orthogonal elements in the closed unit ball of \(E\), infinite subsets \(N \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_{n-1} \supsetneq N_n \supsetneq \cdots\) and a strictly increasing mapping \(\sigma: \mathbb{N} \to \mathbb{N}\) such that for each natural \(n\) there exists a w.u.c. series \(\sum_{k \in N_n} z_{n,k}\) in \(E\) with \(\|z_{n,k}\| \leq 1\),

\[
|\varphi_{\sigma(i)}(a_i)| > \frac{3}{8} \theta, \quad i = 1, \ldots, n,
\]

and

\[
|\varphi_k(z_{n,k})| > \theta - \frac{336}{\theta} \sum_{j=1}^{n} \delta_j > \frac{\theta}{2}, \quad k \in N_n.
\]

To define \(a_1\), choose \(j_1 \in \mathbb{N}\) with \(\frac{1}{j_1} < \frac{1}{C^2}\delta_1^2\), where \(C\) is the positive constant associated to the w.u.c. series \(\sum_{n \geq 1} z_n\) (see comments in the Introduction).
Since every Hilbert space is of cotype 2 (compare [25, page 32]) we have

\[
\frac{1}{j_1} \sum_{k=1}^{j_1} \|z_k\|_{\varphi_m}^2 \leq \frac{1}{j_1} \int_D \left\| \sum_{k=1}^{j_1} \varepsilon_k z_k \right\|_{\varphi_m}^2 \, d\mu
\]

\[
\leq \frac{1}{j_1} \int_D \|\varphi_m\| \left\| \sum_{k=1}^{j_1} \varepsilon_k z_k \right\|_{\varphi_m}^2 \, d\mu \leq \frac{C^2}{j_1} < \delta_1^2,
\]

where \( m \in \mathbb{N} \), \( D = \{-1, 1\}^n \), \( \varepsilon_k \in \{\pm 1\} \) and \( \mu \) is the uniform probability measure on \( D \). Since the above inequality is satisfied for every \( m \in \mathbb{N} \), there exist \( \sigma(1) \in \{1, \ldots, j_1\} \) and an infinite subset \( N_1 \subset \mathbb{N} \) such that \( \sigma(1) < \min N_1 \) and \( \|z_{\sigma(1)}\|_{\varphi_m} < \delta_1 \), for every \( m \in N_1 \).

Applying Lemma 2.2 to \( z_{\sigma(1)} \) and \( \{\varphi_{\sigma(1)}\} \cup \{\varphi_m\}_{m \in N_1} \) we obtain two elements \( a_1, y_1 \) in the closed unit ball of \( E_{\sigma(1)} \) and two tripotents \( u_1, v_1 \in E^{**} \) such that \( a_1 \leq u_1 \leq y_1 \leq v_1 \).

\[ |\varphi_{\sigma(1)}(a_1)| > \frac{3}{4} \theta > \frac{3}{8} \theta, \quad \text{and} \quad \|v_1\|_{\varphi_m} < \frac{8}{\theta} \delta_1 < \frac{16}{\theta} \delta_1, \quad m \in N_1. \]

We define \( z_{1,k} := B(y_1, y_1)z_k \), \( k \in N_1 \), which are elements in the closed unit ball of \( E \) by Lemma 1.2. Clearly \( \sum_{k \in N_1} z_{1,k} \) also is a w.u.c. series. Lemma 1.2 assures that \( z_{1,k} \) is contained in \( E \cap E_0^{**}(u_1) \). Since \( a_1 \in E_2^{**}(u_1) \), we deduce that \( a_1 \perp z_{1,k}, \forall k \in \mathbb{N} \) (compare with the reformulations of orthogonality given in page 312). Moreover \( \left\| \sum_{k \in \mathcal{F}} \varepsilon_k z_{1,k} \right\| = \left\| B(y_1, y_1) \left( \sum_{k \in \mathcal{F}} \varepsilon_k z_k \right) \right\| \leq C, \quad \text{for every finite } \mathcal{F} \in N_1 \) and \( |\varepsilon_k| \) in \( C \). Now, noticing that \( y_1 \in E_2^{**}(v_1) \), Lemma 2.1 applies to assure that

\[ |\varphi_k(z_{1,k})| \geq |\varphi_k(z_k)| - |\varphi_k(z_k - z_{1,k})| > \theta - 21 \frac{16}{\theta} \delta_1 \left( \frac{\theta}{2} \right), \]

for all \( k \in N_1 \).

Suppose now, in our inductive step, that \( a_1, \ldots, a_n, N_n \subset N_{n-1} \subset \cdots \subset N_1 \subset \mathbb{N}, \sigma(1) < \sigma(2) < \cdots < \sigma(n) \), and the w.u.c. series \( \sum_{k \in L_{n-1}} z_{n,k} \) in \( E \) have been constructed satisfying the corresponding induction hypothesis.

Take \( j_{n+1} \in \mathbb{N} \) with \( \frac{1}{j_{n+1}} < \frac{1}{C} \delta_{n+1}^2 \) and a subset \( D_{n+1} \subset N_n \) with exactly \( j_{n+1} \) elements. As before, for \( m \in N_n \) we have

\[
\frac{1}{j_{n+1}} \sum_{k \in D_{n+1}} \|z_{n,k}\|_{\varphi_m}^2 \leq \frac{1}{j_{n+1}} \int_D \left\| \sum_{k \in D_{n+1}} \varepsilon_k z_{n,k} \right\|_{\varphi_m}^2 \, d\mu
\]

\[
\leq \frac{1}{j_{n+1}} \int_D \|\varphi_m\| \left\| \sum_{k \in D_{n+1}} \varepsilon_k z_{n,k} \right\|_{\varphi_m}^2 \, d\mu \leq \frac{C^2}{j_{n+1}} < \delta_{n+1}^2,
\]
hence there exist $\sigma(n+1) \in D_{n+1}$ and an infinite subset $N_{n+1} \subseteq N_n$ such that $\sigma(n+1) < \min N_{n+1}$ and $\|z_{n,\sigma(n+1)}\|_{\varphi_m} < \delta_{n+1}$, for every $m \in N_{n+1}$.

Applying Lemma 2.2 to $z_{n,\sigma(n+1)}$ and $\{\varphi_{\sigma(n+1)}\} \cup \{\varphi_m\}_{m \in N_{n+1}}$ we obtain two elements $a_{n+1}, y_{n+1}$ in the closed unit ball of $E_{z_{n,\sigma(n+1)}}$ and two tripotents $u_{n+1}, v_{n+1} \in (E_{z_{n,\sigma(n+1)}})^{\ast\ast}$ such that $a_{n+1} \leq u_{n+1} \leq y_{n+1} \leq v_{n+1}$,

$$|\varphi_{\sigma(n+1)}(a_{n+1})| > \frac{3}{8} \theta, \quad \text{and} \quad \|v_{n+1}\|_{\varphi_m} < \frac{16}{\theta} \delta_{n+1}, \quad m \in N_{n+1}.$$  

By the induction hypothesis, $z_{n,k} \perp a_j$, for all $j \in \{1, \ldots, n\}$, $k \in N_n$. Since $a_{n+1}, y_{n+1}, u_{n+1}$, and $v_{n+1}$ belong to $(E_{z_{n,\sigma(n+1)}})^{\ast\ast}$, the equivalent reformulations of orthogonality given in page 312, guarantee that they are all orthogonal to $a_j$, for all $j \in \{1, \ldots, n\}$.

We define $z_{n+1,k} := B(y_{n+1}, y_{n+1})(z_{n,k})$, $k \in N_{n+1}$. Again, Lemma 1.2 assures that $z_{n+1,k}$ is contained in $E \cap E_0^{\ast\ast}(u_{n+1})$. Since $a_{n+1} \in E_2^{\ast\ast}(u_{n+1})$, we deduce that $a_{n+1}$ is orthogonal to each $z_{n+1,k}$, $\forall k \in N_{n+1}$. Since $y_{n+1}$ and $z_{n,k}$ are orthogonal to $a_j$ for all $j \in \{1, \ldots, n\}$, $k \in N_{n+1}$, using (3), it can be seen that

$$z_{n+1,k} = B(y_{n+1}, y_{n+1})(z_{n,k}) = z_{n,k} - 2L(y_{n+1}, y_{n+1})(z_{n,k}) + Q(y_{n+1})^2(z_{n,k})$$

is orthogonal to $a_j$, for all $j \in \{1, \ldots, n\}$, $k \in N_{n+1}$. Moreover,

$$\left\|\sum_{k \in \mathcal{F}} \varepsilon_k z_{n+1,k}\right\| = \left\|B(y_{n+1}, y_{n+1})\left(\sum_{k \in \mathcal{F}} \varepsilon_k z_{n,k}\right)\right\| \leq C,$$

for any finite subset $\mathcal{F} \subset N_{n+1}$, and $|\varepsilon_k| = 1$ in $C$.

Finally, since $y_{n+1} \in E_2^{\ast\ast}(v_{n+1})$, Lemma 2.1 assures that

$$|\varphi_k(z_{n+1,k})| \geq |\varphi_k(z_{n,k})| - |\varphi_k(z_{n,k} - z_{n+1,k})|$$

$$> \theta - \frac{336}{\theta} \sum_{j=1}^n \delta_j - 21 \frac{16}{\theta} \delta_{n+1}$$

$$= \theta - \frac{336}{\theta} \sum_{j=1}^{n+1} \delta_j \left(\frac{\theta}{2}\right) \quad \text{for all} \quad k \in N_{n+1}.$$  

b) $\Rightarrow$ b') Since the elements $(a_n)$ are mutually orthogonal, the subtriple $\mathcal{C}$ generated by the family $\{a_n : n \in N\}$ coincides with the $\ell_{\infty}$-sum $\bigoplus_{n}^\infty E_{a_n}$. We recall that each $E_{a_n}$ is isomorphic to $C_0(L)$, for a suitable locally compact Hausdorff space. Therefore $\mathcal{C}$ is triple-isomorphic to an abelian C*-algebra and the restriction of $K$ to $\mathcal{C}$ cannot be relatively weakly compact.

b') $\Rightarrow$ a) is obvious.
A Dieudonné-type theorem for JC*-triples was established in [11, Theorem 4.2]. When in the proof of the just quoted result, Theorem 2.3 replaces [11, Theorem 3.5], we obtain the following generalization of Dieudonné’s theorem in the more general setting of JB*-triples.

**Theorem 2.4.** Let \((\phi_n)\) be a sequence in the dual of a JB*-triple \(E\) such that the sequence \((\phi_n(r(x)))\) converges whenever \(r(x)\) is the range tripotent of a norm-one element \(x\) in \(E\). Then there exists \(\phi\) in \(E^*\) satisfying that \((\phi_n)\) converges weakly to \(\phi\). In particular, if \((\phi_n(r(x))) \to 0\), for every range tripotent, \(r(x)\), of a norm-one element \(x\) in \(E\), then \((\phi_n)\) is a weakly null sequence in \(E^*\).

The vector-valued version of the above theorem follows now as a consequence. The following corollary also generalizes the main result in [19] with a shorter and simpler proof.

**Corollary 2.5.** Let \(E\) be a JB*-triple, \(X\) a Banach space and \((T_n)\) a sequence of weakly compact operators from \(E\) to \(X\). Suppose that \(\lim T_n^{**}(r(x))\) exists whenever \(r(x)\) is the range tripotent of a norm-one element \(x\) in \(E\). Then there exists a unique weakly compact operator \(T : E \to X\), such that \(T^{**}(z) = \lim T_n^{**}(z)\), for every \(z \in E^{**}\).

**Proof.** We claim that for each \(z \in E^{**}\), \((T_n^{**}(z))\) is a norm convergent sequence. Otherwise, there exist \(z \in E^{**}\), \(\varepsilon > 0\), and \((\sigma(n)) \subset \mathbb{N}\) such that \(\|T_{\sigma(n+1)}^{**}(z) - T_{\sigma(n)}^{**}(z)\| > \varepsilon\), \(\forall n \in \mathbb{N}\). Defining \(S_k = T_{\sigma(k+1)}^{**} - T_{\sigma(k)}^{**}\), we can find norm-one functionals \(\psi_k \in X^*\) satisfying \(|\psi_k(S_k(z))| > \varepsilon\) (\(\forall k \in \mathbb{N}\)). Since \(T_k^{**} : E^{**} \to X^{**}\) is weak*-to-weak* continuous, the sequence \((\psi_k T_k^{**})_{k \in \mathbb{N}}\) lies, in fact, in \(E^{*}\). In particular, the sequence \((\psi_k S_k)\) satisfies, by hypothesis, that \(\lim \psi_k S_k(r) = 0\), for every range tripotent, \(r = r(x)\), of a norm-one element \(x\) in \(E\). Theorem 2.4 assures that \((\psi_k S_k)\) is weakly null in \(E^{*}\), which contradicts \(|\psi_k S_k(z)| = |\psi_k S_k(z)| > \varepsilon\), \((k \in \mathbb{N})\).

The assignment \(z \mapsto L_z := \lim T_n^{**}(z)\) defines a linear mapping \(L : E^{**} \to X^{**}\), which is continuous by the Uniform Boundedness Principle. Since each \(T_n\) is weakly compact we have \(T_n^{**}(E^{**}) \subseteq X\), \(\forall n \in \mathbb{N}\). In particular \(L(E^{**}) \subseteq X\). Therefore \(T := L|_E : E \to X\) is a well-defined bounded linear operator.

Theorem 2.4 implies that, for each \(\psi \in X^*\) the \(\psi T_n^{**} = T_n^{*}(\psi) \in E^*\) converge weakly to some \(\varphi \in E^*\). Thus \(\psi L = \varphi \in E^*\), which proves that \(L\) is weak*-to-weak* continuous. It is now clear that \(T^{**} = L\). Finally, the expression \(T^{**}(E^{**}) = L(E^{**}) \subseteq X\) shows that \(T\) is weakly compact.

**Acknowledgements.** The authors would like to express their gratitude to the anonymous referee whose valuable comments made the presentation more consistent.
REFERENCES


