ON A UNIQUENESS PROPERTY OF SECOND CONVOLUTIONS

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1. Introduction and Main Result

Let $M_\infty$ denote the space of all finite nontrivial complex Borel measures on the real line whose variation has a fast decay at $-\infty$:

\begin{equation}
\int_{-\infty}^{0} e^{r|t|} \, d|\mu(t)| < \infty, \quad \text{for every} \quad r > 0.
\end{equation}

It follows from (1) that the Fourier-Stieltjes transform of every measure $\mu \in M_\infty$,

\[ \hat{\mu}(z) := \int_{-\infty}^{\infty} e^{izt} \, d\mu(t), \]

converges uniformly on compact subsets of the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im} \, z > 0\}$ to a function analytic in $\mathbb{C}_+$. Let $l(\mu) := \inf \supp \, \mu$ denote the left boundary of the support of $\mu$, and $\mu^{n*}$ the $n$th convolution power of $\mu$.

The following uniqueness property of $n$th convolutions of measures from $M_\infty$ was discovered in connection with some probabilistic results (see for example [1], [7], [8], [9], [10] and the literature therein): Let $n \geq 3$ be an integer, and let $\mu \in M_\infty$ be such that $l(\mu) = -\infty$. Then every half-line $(-\infty, a)$, $a \in \mathbb{R}$, is a uniqueness set for the $n$th convolution $\mu^{n*}$, in the sense that the implication holds: Suppose $\nu \in M_\infty$ and

\begin{equation}
\text{there exists} \ a \in \mathbb{R} \ \text{such that} \ \mu^{n*}|_{(-\infty,a)} = \nu^{n*}|_{(-\infty,a)}. \ \text{Then} \ \mu^{n*} = \nu^{n*}.
\end{equation}

It is also known that property (2) does not hold for $n = 2$. An easy way to check this is to take two measures $\xi_1, \xi_2 \in M_\infty$ such that $l(\xi_1 + \xi_2) = -\infty$ and $\xi_1 * \xi_2 = 0$ on some half-line $(-\infty, a)$. Then the measures $\mu = \xi_1 + \xi_2$ and $\nu = \xi_1 - \xi_2$ belong to $M_\infty$, $l(\mu) = -\infty$ and we have

\[ (\mu^{2*} - \nu^{2*})|_{(-\infty,a)} = 4\xi_1 * \xi_2|_{(-\infty,a)} = 0. \]

Received August 15, 2007.
For example, let $\xi_j \in M_\infty$ be the measures with Fourier-Stieltjes transforms

$$\hat{\xi}_j(z) = e^{(-1)/e^{-iz}}, \quad j = 1, 2.$$  

From $\hat{\xi}_1 \hat{\xi}_2 = 1$, we see that $\xi_1 * \xi_2$ is the unit measure concentrated at the origin, so that $(\xi_1 + \xi_2)^{2*} - (\xi_1 - \xi_2)^{2*} = 4\xi_1 * \xi_2 = 0$ on $(-\infty, 0)$.

It turns out that there cannot be more than two different second convolutions which agree on a half-line. The aim of this note is to prove the following

**Theorem 1.** Assume a measure $\mu \in M_\infty$ satisfies $l(\mu) = -\infty$. Suppose there exists $a \in \mathbb{R}$ and measures $\nu, \phi \in M_\infty$ such that

$$\mu^{2*}|(-\infty,a) = \nu^{2*}|(-\infty,a) = \phi^{2*}|(-\infty,a),$$

and $\nu^{2*} \neq \phi^{2*}$. Then either $\nu^{2*} = \mu^{2*}$ or $\phi^{2*} = \mu^{2*}$.

An immediate corollary is the following uniqueness property of the second convolutions:

**Corollary 2.** For every $\mu \in M_\infty$, $l(\mu) = -\infty$, there is a real number $a_0 = a_0(\mu)$ such that $\mu^{2*}$ is uniquely determined by its values on $(-\infty, a)$, $a > a_0$, i.e. if $\nu \in M_\infty$ and there exists $a > a_0$ such that $\mu^{2*}|(-\infty,a) = \nu^{2*}|(-\infty,a)$, then $\mu^{2*} = \nu^{2*}$.

We also mention a uniqueness result for squares of analytic functions:

**Corollary 3.** Assume functions $f$, $g$, and $h$ are analytic in the punctured unit disk $0 < |z| < 1$, and that $f$ has an essential singularity at the origin. Suppose that both functions $f^2 - g^2$ and $f^2 - h^2$ have a pole or a removable singularity at the origin and $g^2 \neq h^2$. Then either $g^2 = f^2$ or $h^2 = f^2$.

This is just a particular case of Theorem 1 for measures concentrated on the set of integers, and follows from it by the change of variable $z = \exp(-it)$.

**2. Remarks**

1. Observe that condition (1) is crucial for the uniqueness property (2): The property (2) does not in general hold for measures whose Fourier–Stieltjes transform is not analytic in $\mathbb{C}_+$, see [7], [8] and [1]. A comprehensive survey of results on this and similar uniqueness properties can be found in [9].

2. As it was observed in [7], the uniqueness property of $n$th convolutions (2) is closely connected with the Titchmarsh convolution theorem and its extensions. The classical Titchmarsh convolution theorem states that if $\xi_1$ and $\xi_2$ are finite Borel measures satisfying $l(\xi_j) > -\infty$, $j = 1, 2$, then $l(\xi_1 * \xi_2) = l(\xi_1) + l(\xi_2)$. This is not true for measures with unbounded support,
for there exist measures $\xi_j$, $j = 1, 2$, $l(\xi_1) = -\infty$, such that $l(\xi_1 * \xi_2) > -\infty$. Such measures can be taken from $M_\infty$, see example (3). However, it was shown in [8] that the conclusion of Titchmarsh convolution theorem holds true whenever the variation of measures satisfies a condition at $-\infty$ more restrictive than (1):

$$\int_{-\infty}^{0} e^{r|t|\log|t|} d|\mu(t)| < \infty, \text{ for every } r > 0. \tag{5}$$

Second convolutions of such measures enjoy the uniqueness property above ([7], [8]). Moreover, examples similar to (3) show that restriction (5) cannot be weakened. Analogous results for unbounded measures were established in [2].

Observe that extensions of the Titchmarsh convolution theorem have also applications in the theory of invariant subspaces, see [2], [3] and [4].

3. The Titchmarsh convolution theorem has been extended to linearly dependent measures: the equality

$$l (\xi_1 * \cdots * \xi_n) = \sum_{j=1}^{n} l(\xi_j)$$

holds for linearly dependent measures $\xi_j \in M_\infty$, $j = 1, \ldots, n$, $n \geq 3$, in “general position”, for the precise statement see [5]. Our proof of Theorem 1 below is a fairly easy consequence of this result.

3. Proof of Theorem 1

The following lemma is a particular case of Theorem 4 in [5]:

**Lemma 4.** (i) Suppose measures $\xi_1, \xi_2, \xi_3 \in M_\infty$ are linearly independent over $\mathbb{C}$. Then

$$l(\xi_1 * \xi_2 * \xi_3 * (\xi_1 + \xi_2 + \xi_3)) = l(\xi_1) + l(\xi_2) + l(\xi_3) + l(\xi_1 + \xi_2 + \xi_3). \tag{6}$$

(ii) Suppose measures $\xi_1, \xi_2 \in M_\infty$ are linearly independent over $\mathbb{C}$ and $|a_1| + |a_2| \neq 0$. Then

$$l (\xi_1 * \xi_2 * (\xi_1 + \xi_2) * (a_1\xi_1 + a_2\xi_2)) = l(\xi_1) + l(\xi_2) + l(\xi_1 + \xi_2) + l(a_1\xi_1 + a_2\xi_2).$$

For the convenience of the reader, we recall shortly the main ideas of the proof in [5]. To prove, say (6), by the Titchmarsh convolution theorem, it
suffices to verify the implication

\[ l(\xi_1 \ast \xi_2 \ast \xi_3 \ast (\xi_1 + \xi_2 + \xi_3)) > -\infty \Rightarrow l(\xi_j) > -\infty, \quad j = 1, 2, 3. \]

We may assume that \(\xi_1 \ast \xi_2 \ast \xi_3 \ast (\xi_1 + \xi_2 + \xi_3) = 0\) on \((-\infty, 0)\), so that the product of the Fourier-Stieltjes transforms \(\hat{\xi}_1 \hat{\xi}_2 \hat{\xi}_3 (\hat{\xi}_1 + \hat{\xi}_2 + \hat{\xi}_3)\) belongs to the Hardy space \(H^\infty(C_+)\). Hence, the zero set of the product, and so the zero set of each factor satisfies the Blaschke condition. Now one can use the following argument: If functions \(f_j, j = 1, \ldots, n, n \geq 2\), are analytic in the unit disk, linearly independent and such that the zeros of each \(f_j\) and the sum \(f_1 + \cdots + f_n\) satisfy the Blaschke condition in the disk, then each \(f_j\) must have “slow” growth in the disk. A sharp statement follows from H. Cartan’s second main theorem for analytic curves, see Theorem D in [5]. This argument proves that the growth of each \(\hat{\xi}_j\) in \(C_+\) must satisfy a certain restriction. Next, we have additional information that each function \(\hat{\xi}_j\) is bounded in every horizontal strip in \(C_+\). This allows one to improve the previous estimate to show that numbers \(b_j\) exist such that \(\hat{\xi}_j(z) \exp(ib_jz) \in H^\infty(C_+), j = 1, 2, 3\). This means that \(l(\xi_j) \geq -b_j > -\infty, j = 1, 2, 3\).

We shall also need a simple lemma:

**Lemma 5.** Suppose \(\mu \in M_\infty\) is such that \(l(\mu^{2*}) > -\infty\). Then \(l(\mu) > -\infty\).

Indeed, we may assume that \(\mu^{2*} = 0\) on \((-\infty, 0)\), so that \((\hat{\mu})^2 \in H^\infty(C_+)\). Since \(\hat{\mu}\) is analytic in \(C_+\), we obtain \(\hat{\mu} \in H^\infty(C_+)\). Consider now convolutions \(\mu \ast p_n\), where \(p_n\) is any sequence of smooth functions concentrated on \([0, \infty]\) which converges weakly to the delta-function concentrated at the origin. We have \(\hat{p}_n \hat{\mu} \in (H^\infty \cap H^1)(C_+)\). A standard argument involving inverse Fourier transform along the line \(\text{Im } z = y\) as \(y \to \infty\), proves that \(l(\mu \ast p_n) \geq 0\). Taking the limit as \(n \to \infty\), we conclude that \(l(\mu) \geq 0\).

**Proof of Theorem 1.** Suppose measures \(\mu, \nu, \phi \in M_\infty, l(\mu) = -\infty\), satisfy (4) for some \(a \in R\), and \(\nu^{2*} \neq \phi^{2*}\). Set \(\xi_1 := (\mu + \nu)/2, \xi_2 := (\mu - \nu)/2, \) and \(\eta_1 := (\mu + \phi)/2, \eta_2 := (\mu - \phi)/2\). To prove the theorem, it suffices to show that one of the measures \(\xi_j, \eta_j, j = 1, 2\), is trivial.

Let us assume that it is not so, and show that this leads to a contradiction. Since

\[
(\mu^{2*} - \nu^{2*})|_{(-\infty, a)} = 4\xi_1 \ast \xi_2|_{(-\infty, a)} = 0, \\
(\mu^{2*} - \phi^{2*})|_{(-\infty, a)} = 4\eta_1 \ast \eta_2|_{(-\infty, a)} = 0,
\]

we have

\[
(7) \quad l(\xi_1 \ast \xi_2) > -\infty, \quad l(\eta_1 \ast \eta_2) > -\infty.
\]
Let us show that (7) implies \( l(\mu) > -\infty \), which contradicts the assumption \( l(\mu) = -\infty \).

We shall consider several cases. First, assume that \( \xi_1 \) and \( \xi_2 \) are linearly dependent. Then \( \mu = \xi_1 + \xi_2 = (1 + b)\xi_2 \), for some \( b \in \mathbb{C} \), \( b \neq 0 \), and so

\[
\mu^{2*} = (1 + b)^2 \xi_2^{2*} = \frac{(1 + b)^2}{b} \xi_1 \star \xi_2.
\]

By (7), this gives \( l(\mu^{2*}) > -\infty \). Lemma 5 yields \( l(\mu) > -\infty \).

Assume now that \( \xi_1 \) and \( \xi_2 \) are linearly independent. From \( \mu = \xi_1 + \xi_2 = \eta_1 + \eta_2 \) we have \( \eta_2 = \xi_1 + \xi_2 - \eta_1 \). Now (7) gives

\[-\infty < l(\xi_1 \star \xi_2 \star \eta_1 \star \eta_2) = l(\xi_1 \star \xi_2 \star \eta_1 \star (\xi_1 + \xi_2 - \eta_1)).\]

If \( \xi_1, \xi_2 \) and \( \eta_1 \) are linearly independent, then by part (i) of Lemma 4, we obtain \( l(\xi_j) > -\infty \), \( j = 1, 2 \), and so \( l(\mu) > -\infty \). If \( \xi_1, \xi_2 \) and \( \eta_1 \) are linearly dependent, we have \( \eta_1 = c_1 \xi_1 + c_2 \xi_2 \), for some \( c_1, c_2 \in \mathbb{C} \). Hence,

\[-\infty < l(\xi_1 \star \xi_2 \star \eta_1 \star \eta_2) = l(\xi_1 \star \xi_2 \star (c_1 \xi_1 + c_2 \xi_2)) \star ((1-c_1)\xi_1 + (1-c_2)\xi_2).\]

If either \( c_j \neq 0, j = 1, 2 \), or \( 1 - c_j \neq 0, j = 1, 2 \), then part (ii) of Lemma 4 implies \( l(\xi_j) > -\infty \), and so \( l(\mu) > -\infty \). Otherwise, we may assume that \( c_1 = 0 \) and \( 1 - c_2 = 0 \). This gives

\[-\infty < l(\xi_1 \star \xi_2 \star \eta_1 \star \eta_2) = l(\xi_1^{2*} \star \xi_2^{2*}).\]

From (7) and Lemma 5 we conclude that \( l(\xi_j) > -\infty \), \( j = 1, 2 \), which shows that \( l(\mu) > -\infty \).

REFERENCES


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