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CUP-PRODUCT FOR LEIBNIZ COHOMOLOGY AND DUAL LEIBNIZ ALGEBRAS

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For any Lie algebra $\mathfrak{g}$ there is a notion of *Leibniz cohomology* $HL^*(\mathfrak{g})$, which is defined like the classical Lie cohomology, but with the nth tensor product $\mathfrak{g}^{\otimes n}$ in place of the nth exterior product $\Lambda^n\mathfrak{g}$. This Leibniz cohomology is defined on a larger class of algebras: the *Leibniz algebras* (cf. [L1], [L2]). A Leibniz algebra is a vector space equipped with a product satisfying a variation of the Jacobi identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]. \tag{1}$$

The purpose of this paper is to construct, by means of shuffles, a cup-product on the cohomology groups $HL^*(\mathfrak{g})$ of the Leibniz algebra $\mathfrak{g}$:

$$\cup\colon HL^p(\mathfrak{g}) \times HL^q(\mathfrak{g}) \to HL^{p+q}(\mathfrak{g}).$$

This product is not associative nor commutative, but satisfies the formula

$$(f \cup g) \cup h = f \cup (g \cup h) + (-1)^{|h||g|} f \cup (h \cup g). \tag{2}$$

This property of Leibniz cohomology has been obtained independently and by a different method by J.-M. Oudom [O].

It turns out that this relation is, in a certain sense, dual to relation (1).

Here are two features of this duality. First, for any algebra $\mathfrak{g}$ satisfying relation (1) (i.e. Leibniz algebra) and any algebra R satisfying a non-graded version of (2) (i.e. dual Leibniz algebra) the tensor product $\mathfrak{g} \otimes R$ can be equipped with a structure of Lie algebra. Second, let R be as above and define a new product $*$ by the formula

$$r * r' = r \cup r' + r' \cup r.$$

Then $(R, *)$ becomes an associative and commutative algebra. Applied to $HL^*(\mathfrak{g})$ it gives precisely the product devised by C. Cuvier in [C].

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Notation. The ground field is denoted by k. Any tensor product over k is denoted by $\otimes$.

1. Leibniz algebras and dual Leibniz algebras.

1.1. *Definition* [L1]. A *Leibniz algebra* $\mathfrak{g}$ is a k-vector space equipped with a bilinear map

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the (right) *Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \forall x,y,z \in \mathfrak{g}. \tag{1.1}$$

Lie algebras are examples of Leibniz algebras since the Leibniz identity is equivalent to the Jacobi identity when the bracket is skew-symmetric. For other examples of Leibniz algebras see [L2], [L-P].

1.2. *Definition.* A *dual Leibniz algebra* R is a k-vector space equipped with a bilinear map

$$(-\,-)\colon R \times R \to R$$

satisfying the (left) relation

$$((rs)t) = (r(st)) + (r(ts)), \quad \forall r,s,t \in R. \tag{1.2}$$

If R is a graded vector space, then there is a notion of graded dual Leibniz algebra:

$$((rs)t) = (r(st)) + (-1)^{|t||s|}(r(ts)).$$

An example of dual Leibniz algebra is given in 1.8.

1.3. PROPOSITION. *Let* $\mathfrak{g}$ *be a Leibniz algebra and let* R *be a dual Leibniz algebra. Then the tensor product* $\mathfrak{g} \otimes R$ *equipped with the bracket*

$$[x \otimes r, y \otimes s] := [x,y] \otimes (rs) - [y,x] \otimes (sr)$$

is a Lie algebra.

PROOF. Skew-symmetry is obvious by definition. Let us show the Jacobi identity, or, equivalently, the relation

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\sigma[x \otimes r, [y \otimes s, z \otimes t]] = 0,$$

where the symmetric group S_3 is acting similarly on both sets x, y, z and r, s, t. This amounts to prove that

$$\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\sigma([x,[y,z]] \otimes (r(st)) - [[y,z],x] \otimes ((st)r)$$
$$- [x,[z,y]] \otimes (r(ts)) + [[z,y],x] \otimes ((ts)r)) = 0,$$

or equivalently

$$\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\sigma(([[x,y],z] - [[x,z],y]) \otimes (r(st)) - [[y,z],x] \otimes ((st)r)$$
$$-([[x,z],y] - [[x,y],z]) \otimes (r(ts)) + [[z,y],x] \otimes ((ts)r)) = 0$$

by the Leibniz relation.

Since the sum is symmetric under the action of S_3, in order to prove that it is 0 it is sufficient to show that the coefficient of $[[x,y],z]$ is 0. This coefficient is

$$(r(st)) + (r(ts)) - ((rs)t) + (r(st)) + (r(ts)) - ((rs)t)$$
$$= 2((r(st)) + (r(ts)) - ((rs)t)) = 0$$

by the dual Leibniz relation.

1.4. Remark. The Leibniz algebras form a Koszul operad in the sense of Ginzburg and Kapranov [G-K]. Proposition 1.3 shows that the notion of dual Leibniz algebra defined here is precisely the dual operad of Leibniz algebras in their sense.

1.5. Proposition. *Let R be a dual Leibniz algebra. Define a new product $*$ on R by symmetrization:*

$$r * s = (rs) + (sr).$$

Then $(R, *)$ is an associative and commutative algebra.

Proof. Only associativity needs to be checked:

$$\begin{aligned} r*(s*t) &= r(st+ts) + (st+ts)r \\ &= r(st+ts) + s(tr+rt) + t(sr+rs), \\ (r*s)*t &= (rs+sr)t + t(rs+sr) \\ &= r(st+ts) + s(rt+tr) + t(rs+sr). \end{aligned}$$

1.6. Remark. Under the Koszul duality of Ginzburg and Kapranov the operad of Lie algebras is dual to the operad of associative and commutative algebras. The symmetrization of proposition 1.5 defines a functor (dual Leibniz) $\to$ (Com) between categories of algebras, which is dual to the inclusion functor (Lie) $\to$ (Leibniz).

1.7. *Shuffles*. Let S_n be the symmetric group considered as the automorphism group of $\{1, 2, \ldots, n\}$. The permutation $\sigma \in S_n$ is a (p, q)-*shuffle* if $p + q = n$ and

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

Recall that the number of (p, q)-shuffles is $\binom{p+q}{p}$.

In the group algebra $k[S_n]$ we define

$$\mathrm{sh}_{pq} := \sum_{\sigma = (p,q)-\text{shuffle}} \sigma.$$

For any vector space V we let $\sigma \in S_n$ act on $V^{\otimes n}$ on the left by place-permutation:

$$\sigma(v_1 \ldots v_n) = (v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(n)}).$$

Here the generator $v_1 \otimes \ldots \otimes v_n$ of $V^{\otimes n}$ is denoted by $v_1 \ldots v_n$. If $\sigma'\sigma$ denotes the product in $k[S_n]$ of the two permutations σ' and σ (i.e. composite in the group S_n), then

$$\sigma'\sigma(v_1 \ldots v_n) = \sigma'(\sigma(v_1 \ldots v_n)).$$

Consider the tensor module of V

$$\bar{T}(V) := V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots.$$

1.8. PROPOSITION. *The product on $\bar{T}(V)$ induced by*

$$(v_1 \ldots v_p) \cdot (v_{p+1} \ldots v_{p+q}) := (1_1 \otimes \mathrm{sh}_{p-1q})(v_1 \ldots v_{p+q})$$

satisfies the dual Leibniz relation (1.2). *In fact $(\bar{T}(V), .)$ is the free dual Leibniz algebra over V. Its associative and commutative algebra* (cf. 1.5) *is the shuffle algebra.*

PROOF. The point is to prove formula (1.2) for the generators $x = v_1 \ldots v_p$, $y = v_{p+1} \ldots v_{p+q}$ and $z = v_{p+q+1} \ldots v_{p+q+r}$. Let τ_{qr} be the permutation such that

$$\tau_{qr}(yz) = zy.$$

It is sufficient to show that

$$(1.8.1) \qquad (1_1 \otimes \mathrm{sh}_{p+q-1r})(1_1 \otimes \mathrm{sh}_{p-1q} \otimes 1_r)$$
$$= (1_1 \otimes \mathrm{sh}_{p-1q+r})(1_{p+1} \otimes \mathrm{sh}_{q-1r} + (1_p \otimes 1_1 \otimes \mathrm{sh}_{r-1q})(1_p \otimes \tau_{qr})).$$

Since the shuffle is an associative operation, we know that

$$\mathrm{sh}_{p+q-1r} \circ (\mathrm{sh}_{p-1q} \otimes 1_r) = \mathrm{sh}_{p-1q+r} \circ (1_{p-1} \otimes \mathrm{sh}_{qr}).$$

So we need only to show that

(1.8.2) $$\mathrm{sh}_{qr} = 1_1 \otimes \mathrm{sh}_{q-1r} + (1_1 \otimes \mathrm{sh}_{r-1q}) \circ \tau_{qr}.$$

This last relation is obvious since it simply says that among the (q,r)-shuffles of $(v_1 \ldots v_{q+r})$ there are two types: those which begin with v_1 and those which begin with v_{q+1}.

Let us now show that $\bar{T}(V)$, equipped with this product, is the free dual Leibniz algebra on V. Let R be a dual Leibniz algebra and $f\colon V \to R$ a linear map. Any generator $x = v_1 \ldots v_p$ in $\bar{T}(V)$ is in fact equal to the iterated product $(v_1 \cdot (\ldots (v_{p-1} \cdot v_p)\ldots))$. So the unique possible extension of f ro $\bar{T}(V)$ is given by $f(x) = (f(v_1)\cdot(\ldots(f(v_{p-1})\cdot f(v_p))\ldots))$. Let us show that $f(x \cdot y) = f(x)\cdot f(y)$ for $x = v_1 \ldots v_p$ and $y = v_{p+1} \ldots v_{p+q}$.

It is clear for $p = 1$, any q. For $q = 1$ one can use relation (1.2) and prove it for any p by induction on p. Again the general case is proved via relation (1.2) by induction on p (write $x = x' \cdot v_p$ and $x \cdot y = x' \cdot ((v_p \cdot y) + (y \cdot v_p))$).

Finally the symmetrization of the product is simply given by $(v_1 \ldots v_q) * (v_{q+1} \ldots v_{q+r}) = \mathrm{sh}_{qr}(v_1 \ldots v_{q+r})$ because of formula (1.8.2).

2. Leibniz cohomology and the cup-product.

2.1. *Leibniz cohomology* [L1]. – Let $\mathfrak{g}$ be a Leibniz algebra, and let A be an associative and commutative algebra over k. The product in A is denoted by $\mu\colon A \otimes A \to A$.

Put $C^n(\mathfrak{g}, A) = \mathrm{Hom}_k(\mathfrak{g}^{\otimes n}, A)$. There is defined a map $\delta\colon C^n(\mathfrak{g}, A) \to C^{n+1}(\mathfrak{g}, A)$ by $\delta(f) = (-1)^{|f|+1} f \circ d$, where $d\colon \mathfrak{g}^{\otimes n+1} \to \mathfrak{g}^{\otimes n}$ is given by

$$d(x_1, \ldots, x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^j (x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

One can show that $d^2 = 0$ (cf. [L1]), therefore $\delta^2 = 0$ and $(C^*(\mathfrak{g}, A), \delta)$ is a cochain-complex. Its homology groups are called the *Leibniz cohomology groups* of $\mathfrak{g}$ with coefficients in A and denoted $\mathrm{HL}^*(\mathfrak{g}, A)$.

2.2. *The cup-product.* The linear map from $k[S_n]$ to itself induced by $\sigma \mapsto \mathrm{sgn}(\sigma)\sigma^{-1}$ for $\sigma \in S_n$ is an anti-homomorphism. The image of $\alpha \in k[S_n]$ under this map is denoted $\tilde{\alpha}$. For any non-negative integers p and q we define a linear map

$$\rho_{pq} := 1_1 \otimes \widetilde{\mathrm{sh}}_{p-1q} = \mathfrak{g}^{\otimes p+q} \to \mathfrak{g}^{\otimes p+q}.$$

We adopt the notational convention of the previous section, henceforth

$$\rho_{pq}(x_1, \ldots, x_{p+q}) = \sum_{\sigma = (p-1,q)-\mathrm{shuffle}} \mathrm{sgn}(\sigma)(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(p+q)}).$$

2.3. THEOREM. *Let $\mathfrak{g}$ be a Leibniz algebra and A an associative and commutative algebra considered as a trivial $\mathfrak{g}$-module. For $f \in C^p(\mathfrak{g}, A)$, $p > 0$, and $g \in C^q(\mathfrak{g}, A)$,*

$q > 0$, the map $(f, g) \mapsto f \cup g := (-1)^{pq}\mu \circ (f \otimes g) \circ \rho_{pq}$ determines a well-defined bilinear map

$$\cup\colon \mathrm{HL}^p(\mathfrak{g}, A) \times \mathrm{HL}^q(\mathfrak{g}, A) \to \mathrm{HL}^{p+q}(\mathfrak{g}, A),$$

called the cup-product.

This cup-product satisfies the formula

$$(x \cup y) \cup z = x \cup (y \cup z) + (-1)^{|y||z|} x \cup (z \cup y),$$

for $x \in \mathrm{HL}^p$, $y \in \mathrm{HL}^q$, $z \in \mathrm{HL}^r$. So $\mathrm{HL}^*(\mathfrak{g}, A)$ is a graded dual Leibniz algebra.

2.4. Corollary. *On $\mathrm{HL}^*(\mathfrak{g}, A)$ the product*

$$x \cdot y = x \cup y + (-1)^{|x||y|} y \cup x$$

makes $\mathrm{HL}^*(\mathfrak{g}, A)$ into an associative and graded commutative algebra. This product is the same as the product defined by C. Cuvier in [C].

2.5. Remark. If $\mathfrak{g}$ is a Lie algebra, then there is a canonical linear map $H^*(\mathfrak{g}, A) \to \mathrm{HL}^*(\mathfrak{g}, A)$ (cf. [L1]), which is an associative algebra map.

Proof of theorem 2.3. The first part of the theorem consists in proving that the cup-product is well-defined. This would be a consequence of the formula

$$\delta(f \cup g) = \delta(f) \cup g + (-1)^{|f|} f \cup \delta(g).$$

Under the definition of the cup-product and the formula $\delta(f) = (-1)^{|f|+1} f \circ d$, one is left to show the following

2.6. Lemma. $\rho_{pq} d_{p+q+1} = (d_{p+1} \otimes 1_q)\rho_{p+1q} + (-1)^p (1_p \otimes d_{q+1})\rho_{pq+1}$.

Proof. Each side of the equality applied to $(x_1, \ldots, x_{p+q+1})$ gives a sum of elements of the form

$$\pm(x_{i_1}, \ldots, x_{i_{k-1}}, [x_{i_k}, x_{i_{k+1}}], x_{i_{k+2}}, \ldots, x_{i_{p+q+1}}),$$

where $i_1 = 1$ and $i_k < i_{k+1}$. In both cases the sign in front of it is $(-1)^{i_{k+1}} \operatorname{sgn}(i)$. The number of terms on the left side is

$$\binom{p+q-1}{p-1} \times \frac{(p+q)(p+q+1)}{2} = \frac{(p+q+1)!}{2(p-1)!q!}$$

and on the right side

$$\frac{p(p+1)}{2}\binom{p+q}{p} + \frac{q(q+1)}{2}\binom{p+q}{p-1}$$
$$= \frac{1}{2}\frac{(p+q)!}{(p-1)!q!}\left[\frac{p(p+1)}{p} + \frac{q(q+1)}{q+1}\right] = \frac{(p+q+1)!}{2(p-1)!q!}.$$

Since the number of terms is the same, it suffices to check that any term appearing on the right hand side belongs to the set of elements appearing on the left hand side.

Let us introduce the operator $\partial_i^j\colon \mathfrak{g}^{\otimes n} \to \mathfrak{g}^{\otimes n-1}$ for $1 \leqq i < j \leqq n$:

$$\partial_i^j(x_1,\ldots,x_n) := (x_1,\ldots,x_{i-1},[x_i,x_j],x_{i+1},\ldots,\hat{x}_j,\ldots,x_n),$$

so that $d_n = \sum_{1 \leqq i < j \leqq n} (-1)^j \partial_i^j$.

1st *case*. Consider $\partial_k^l(1_1 \otimes \sigma^{-1})$ where $1 \leqq k < l \leqq p+1$ and σ is a (p,q)-shuffle acting on $\{2,\ldots,p+q+1\}$. This operator is part of $(d_{p+1} \otimes 1_q)\rho_{p+1q}$. By hypothesis one has $\sigma(k) < \sigma(l)$, therefore $\partial_k^l(1_1 \otimes \sigma^{-1})$ is equal to $(1_1 \otimes \omega^{-1})\partial_{\sigma(k)}^{\sigma(l)}$ for some permutation ω. Since σ is a (p,q)-shuffle and since $l \leqq p+1$, ω is a $(p-1,q)$-shuffle. Indeed the (p,q)-shuffle σ applied to the sequence $\{2,\ldots,p+q+1\}$ gives $\{\sigma(2) < \cdots < \sigma(p+1),\ \sigma(p+2) < \cdots < \sigma(p+q+1)\}$. If one deletes $\sigma(l)$ from this sequence, then the new sequence can be obtained by first deleting $\sigma(l)$ and then applying a $(p-1,q)$-shuffle (acting on $\{2,\ldots,\hat{l},\ldots,p+q+1\}$) since $l \leqq p+1$. So $(1_1 \otimes \omega^{-1})\partial_{\sigma(k)}^{\sigma(l)}$ is part of $\rho_{pq}d_{p+q+1}$.

2nd *case*. Consider $\partial_k^l(1_1 \otimes \sigma^{-1})$ where $p+1 \leqq k < l \leqq p+q+1$ and σ is a $(p-1,q+1)$-shuffle acting on $\{2,\ldots,p+q+1\}$. This operator is part of $(1_p \otimes d_{q+1})\rho_{pq+1}$. By hypothesis one has $\sigma(k) < \sigma(l)$, therefore $\partial_k^l(1_1 \otimes \sigma^{-1})$ is equal to $(1_1 \otimes \omega^{-1})\partial_{\sigma(k)}^{\sigma(l)}$ for some permutation ω. Since σ is a $(p-1,q+1)$-shuffle and $k \geqq p+1$, ω is a $(p-1,q)$-shuffle and so $(1_1 \otimes \omega^{-1})\partial_{\sigma(k)}^{\sigma(l)}$ is part of $\rho_{pq}\mathrm{d}_{p+q+1}$.

End of the proof of theorem 2.3. Let us now show that $HL^*(\mathfrak{g}, A)$ is a dual Leibniz algebra. Let f, g and h be cocycles representing x, y, and z respectively. By definition

$$h \cup g = (-1)^{qr}\mu \circ (h \otimes g) \circ \rho_{rq}, \text{ for } h \in C^r(\mathfrak{g}, A) \text{ and } g \in C^q(\mathfrak{q}, A).$$

Since A is commutative we have

$$\mu \circ (h \otimes g) \circ \rho_{rq} = \mu \circ (g \otimes h) \circ \tau_{rq} \circ \rho_{rq}.$$

It follows that the dual Leibniz algebra relation for the cup-product (formula (2) of the introduction) would be a consequence of the relation

$$(\rho_{pq} \otimes 1_r) \circ \rho_{p+qr} = (1_p \otimes \rho_{qr} + (-1)^{rq} 1_p \otimes \tau_{rq}\rho_{rq}) \circ \rho_{pq+r}.$$

Under the anti-homomorphism $\alpha \mapsto \tilde{\alpha}$ this is precisely proposition 1.8 (cf. formula 1.8.1).

Proof of corollary 2.4. The symmetrized product (graded version) makes $HL^*(\mathfrak{g}, A)$ into an associative and commutative algebra by proposition 1.5. From the definition of the cup-product and by formula (1.8.2) we conclude that the symmetrized product is induced by $\widetilde{\mathrm{sh}}_{pq}$ as in [C].

2.7. *Generalization.* – There exists a category ΔS, whose objects are $[n]$ for $n \geqq 0$ and which is made of the simplicial category Δ and the symmetric groups S_n (cf. [L1, chapter 6]). For any functor from ΔS to the category of k-modules one can construct a complex analogous to $C^*(\mathfrak{g}, A)$ described in 2.1 (cf. [L2, §8.2]). Since only the properties of the shuffles with respect to the boundary map d are used in the proof of theorem 2.3 it is clear that it can be extended to this more general setting.

2.8. *Leibniz cohomology of matrices.* – Let $\mathfrak{gl}(A)$ be the Lie algebra of matrices with entries in the associative unitary k-algebra A. When k is a characteristic zero field $\mathrm{HL}_*(\mathfrak{gl}(A))$ has been computed by Cuvier [C] (see also [L1]). Interpreted in the cohomological framework, it gives an isomorphism of graded vector spaces

$$\mathrm{HL}^*(\mathfrak{gl}(A)) \cong T(HH^*(A)[1]),$$

which is in fact an isomorphism of *graded dual Leibniz algebras*, provided that one puts on the right hand side the structure of a free graded dual Leibniz algebra (cf. 1.8).

2.9. Remark. Theorem 2.3 suggests that for any $\mathscr{P}$-algebra A, where $\mathscr{P}$ is a Koszul operad, the graded module $H^*_{\mathscr{P}}(A)$ is a graded $\mathscr{P}^!$-algebra, where $\mathscr{P}^!$ is the dual operad. We will treat this more general case in another paper.

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