ON LAVINE'S FORMULA FOR TIME-DELAY

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Abstract.

Lavine's formula gives a connection between time-delay and potential in scattering theory. A time-dependent proof is given for potentials $V = V_1 + V_2$, $V_1(x) = O(|x|^{-1-\varepsilon})$, $x \cdot \nabla V_1(x) = O(|x|^{-1-\varepsilon})$, $V_2(x) = O(|x|^{-2-\varepsilon})$, as $|x| \to \infty$.

1. Introduction.

The present note is devoted to an essentially time-dependent proof of Lavine's formula for time-delay. It can be viewed as a continuation of [1]. To state the results, let $H_0 = -\Delta$ and $H = H_0 + V$ denote the free and full Hamiltonian, respectively, in $\mathscr{H} = L^2(\mathbb{R}^n)$, with $V(x) = O(|x|^{-\beta})$, $\beta > 1$, as $|x| \to \infty$. For such short range potentials existence and completeness of the wave operators $W_{\pm}$ is well known. Let $S = W_{+}^{*} W_{-}$ denote the scattering operator, and $S = \{ S(\lambda) \}$ its decomposition into scattering matrices in the spectral representation for $H_0$. The Eisenbud–Wigner time-delay operator is defined in this spectral representation by

$$T = \{-iS(\lambda)^{\dagger} (d/d\lambda) S(\lambda)\},$$

see [1]. Let $D = (2i)^{-1}(x \cdot \nabla + V \cdot x)$ denote the generator of dilations. Lavine's expression for time-delay is the right hand side of the following formula, which is the main result obtained here:

$$\langle f, TH_0 g \rangle = \int_{-\infty}^{\infty} \langle e^{-isH} W_- f, (H - i/2[H,D]) e^{-isH} W_- g \rangle \, ds$$

for a dense set of vectors $f, g \in \mathscr{H}$. Formally $H - i/2[H,D] = V + \frac{1}{2} x \cdot \nabla V$, so (1.1) establishes a connection between the potential and time-delay.

Here (1.1) is proved for potentials satisfying $V = V_1 + V_2$, $V_1 \in C^1(\mathbb{R}^n)$,

$$|V_1(x)| + |x \cdot \nabla V_1(x)| \leq c(1 + |x|)^{-1-\varepsilon}, \quad \varepsilon > 0,$$

and

$$V_2(x) = O(|x|^{-2-\varepsilon})$$

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as $|x| \to \infty$; $V_2$ can have some local singularities. The proof given here follows essentially the formal proof given in [5], see also [3]. In this proof technical results on $[H, D]$ and $[e^{-itH}, D]$ first given by Mourre [4] play an important role. The proof given here also shows that the alternative definition of time-delay [5] can be made rigorous, and agrees with the usual one.

The result in Lemma 2.7 might be of independent interest. Here it is shown that the four operators $W_{\pm} \varphi(H_0), W^*_{\pm} \varphi(H)$, map the domain of $D$ into itself. $\varphi$ is a smooth function with compact support in $(0, \infty) \setminus \sigma_p(H)$. $\sigma_p(H)$ denotes the point spectrum of $H$.

In [2] Lavine proved that the right hand side of (1.1) equals an expression involving sojourn times. The result was proved in $L^2(\mathbb{R}^1)$ for $V = V_1, V_1$ satisfying the condition given above. The connection with $T$ was not given in [2]. Combining (1.1) with the results in [1] a connection with sojourn times has been established.

Recently Martin [3] has given an extensive review of time-delay and related topics. See also [3] for applications of (1.1).

This note is a revision of a preliminary version, in which stronger conditions were imposed on $V$. Partly based on this preliminary version Narnhofer [6] has recently discussed (1.1) and related results, using a somewhat different approach, for essentially the same class of potentials as defined above.

II. The results.

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ denote configuration space and $\mathcal{F}$ the Fourier transform. $\mathcal{D}(T)$ denotes the domain of an operator $T$. $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Let $\mathcal{S}'(\mathbb{R}^n)$ denote the tempered distributions. The weighted Sobolev space $H^{m,s} = H^{m,s}(\mathbb{R}^n)$ is given by

$$H^{m,s} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \|f\|_{m,s} = \|(1 + x^2)^{s/2}(1 - \Delta)^{m/2}f\|_{L^2} < \infty \right\}.$$  

The free Hamiltonian is $H_0 = -\Delta$ with $\mathcal{D}(H_0) = H^{2,0}$. Let $L^2(S^{n-1})$ denote the square integrable functions on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. The spectral representation for $H_0$ is given by

$$F: \mathcal{H} \to \mathcal{H}_s = L^2((0, \infty); L^2(S^{m-1}))$$

defined by

$$(Ff)(\lambda)(\omega) = 2^{-1/2}\lambda^{(n-2)/4}(\mathcal{F}f)(\lambda^{1/2}\omega),$$

$\lambda > 0, \omega \in S^{n-1}$. See [1] for further details.

The following short range assumption is imposed on the potential.

ASSUMPTION 2.1. Let $V$ be multiplication by a real-valued function $V(x)$. Let $V(x) = V_1(x) + V_2(x)$, where $V_1$ is continuously differentiable with
\[ |V_1(x)| + |x \cdot \nabla V_1(x)| \leq c(1 + |x|)^{-1-\varepsilon} \]

for some \( \varepsilon > 0 \), \( c > 0 \), and \( V_2 \) satisfies

\[ V_2 : H^{2.0} \to H^{0.\beta} \]

is compact for some \( \beta > 2 \).

\( H = H_0 + V \) is the operator sum. Let \( \sigma_p(H) \) denote the point spectrum for \( H \), and \( E \) the spectral measure for \( H \). \( E_0 \) denotes the spectral measure for \( H_0 \). Under the above assumption existence and completeness of the wave operators

\[ W_\pm = s - \lim\limits_{t \to \pm \infty} e^{itH} e^{-itH_0} \]

is well known, see e.g. [8]. The scattering operator \( S = W_+^* W_- \) is decomposable in \( \mathscr{H} \), viz.

\[(FSf)(\lambda) = S(\lambda)(Ff)(\lambda), \quad \lambda \in (0, \infty) \setminus \sigma_p(H).\]

Usually this is written \( S = \{S(\lambda)\} \). \( S(\lambda) \) is the scattering matrix. If \( V = V_1 \), or \( V = V_2 \), the Eisenbud–Wigner time-delay operator was defined in [1] by

\[ T = \{-iS(\lambda)^*(d/d\lambda)S(\lambda)\} \]

The generalization to \( V = V_1 + V_2 \) follows from the proof of Theorem 3.6 in [1]. Theorem 3.8 in [1] remains valid for this larger class of potentials.

Let \( D = (2i)^{-1} (x \cdot \nabla + \nabla \cdot x) \) denote the generator of dilations. Note that \( i[H_0, D] = 2H_0 \). \([V, D]\) can be defined on \( \mathcal{D}(D) \cap \mathcal{D}(H_0) \) as a quadratic form. Assumption 2.1 implies that \([V, D]\) extends to a bounded operator, denoted \([V, D]^a\), from \( H^{2.0} \) to \( H^{-2.0} \). \([H, D]^a\) is defined similarly, and one has

\[ H - i/2[H, D]^a = V - i/2[V, D]^a \]

as bounded operators from \( H^{2.0} \) to \( H^{-2.0} \). Sometimes it is convenient to use the notation

\[ \tilde{V} = V - i/2[V, D]^a. \]

The main result of this note is the following theorem.

**Theorem 2.2.** Let \( V \) satisfy Assumption 2.1. Let \( [a, b] \subset (0, \infty) \setminus \sigma_p(H) \) be a finite interval, and let \( f, g \in E_0([a, b]) \mathscr{H} \). Then one has

\[
\langle f, TH_0 g \rangle = \int_{-\infty}^{\infty} \langle e^{-isH} W_- f, (H - i/2[H, D]^a) e^{-isH} W_- g \rangle ds.
\]

The proof is based on the following Lemmas.
Lemma 2.3. Let \([a, b] \subset (0, \infty) \setminus \sigma_p(H)\) be a finite interval. There exists \(c > 0\), depending only on \(a\) and \(b\), such that
\[
\int_{-\infty}^{\infty} |\langle e^{-itH}E([a, b])f, Ve^{-itH}E([a, b])g \rangle| \, dt \leq c \|f\| \|g\|
\]
for all \(f, g \in \mathcal{H}\). The same result is true with \(V\) replaced by \(\tilde{V}\).

Proof. Assumption 2.1 implies \(V \in \mathcal{B}(H^{2, -\delta}, H^{0, \delta})\) and \(\tilde{V} \in \mathcal{B}(H^{2, -\delta}, H^{-2, \delta})\) for some \(\delta > 1/2\). The result now follows from well known local smoothness results due to Kato and Lavine, see e.g. [9].

Lemma 2.4. (i) \([D, e^{-itH}]\) extends to a bounded operator from \(H^{2, 0}\) to \(H^{-2, 0}\), which satisfies
\[
\|[D, e^{-itH}]g\|_{\mathcal{B}(H^{1,0}, H^{-2,0})} \leq c(1 + |t|)
\]
for all \(t \in \mathbb{R}\).

(ii) Let \(\varphi \in C_0^\infty(\mathbb{R}^n)\). Then \([D, \varphi(H)]\) extends to a bounded operator from \(H^{-1, 0}\) to \(H^{1, 0}\).

Proof. See [7; Lemma 7.4]. These results extend slightly results due to Mourre [4]. Note that the extension is needed here.

Lemma 2.5. Let \(\varphi \in C_0^\infty((0, \infty) \setminus \sigma_p(H))\). Then one has for all \(t \in \mathbb{R}\)
\[
\|(D + \hat{t})^{-1}e^{-itH}\varphi(H)(D + \hat{t})^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq c(1 + |t|)^{-1}.
\]

Proof. The following commutator is computed on \(\mathcal{D}(D) \cap \mathcal{D}(H)\) as a quadratic form:
\[
[D, e^{-itH}] = e^{-itH}(e^{itH}De^{-itH} - D)
\]
\[
= e^{-itH} \int_0^t e^{isH}[H, D]e^{-isH} \, ds
\]
\[
= e^{-itH}2tH + e^{-itH} \int_0^t e^{isH}(i[V, D] - 2V)e^{-isH} \, ds.
\]
This result now extends as an equality between bounded operators from \(H^{2, 0}\) to \(H^{-2, 0}\). Let \(\varphi \in C_0^\infty((0, \infty) \setminus \sigma_p(H))\) be given, and let \(\chi \in C_0^\infty((0, \infty) \setminus \sigma_p(H))\) be identically one on the support of \(\varphi\). Let \(\psi(\lambda) = \lambda^{-1}\chi(\lambda)\). Then \(\psi \in C_0^\infty\), \(\varphi(H) = H\psi(H)\varphi(H)\), and
\[
e^{-itH} \varphi(H) = \frac{1}{2t} \psi(H) e^{-itH} 2tH \varphi(H)
\]

\[
= \frac{1}{2t} \psi(H)[D, e^{-itH}] \varphi(H) + \frac{1}{t} \psi(H) \int_0^t e^{isH} \Lambda e^{-isH} d\varphi(H).
\]

Lemma 2.3 implies

\[
\left\| \psi(H) \int_0^t e^{isH} \Lambda e^{-isH} d\varphi(H) \right\| \leq c
\]

for all \( t \in \mathbb{R} \). The result now follows using Lemma 2.4 (ii).

**Remark 2.6.** (2.3) and related results were proved in [1] for \( V=0 \). The idea used in handling \([D, e^{-itH}]\) above is due to Mourre [4].

**Lemma 2.7.** Let \( \varphi \in C_0^\infty((0, \infty) \setminus \sigma_p(H)) \). The operators \([D, W_- \varphi(H_0)]\), \([D, W_+ \varphi(H_0)]\), \([D, W^* \varphi(H)]\), and \([D, W^* \varphi(H)]\), defined as quadratic forms on \( \mathcal{D}(D) \times \mathcal{D}(D) \), extend to bounded operators on \( \mathcal{H} \). In particular, all four operators \( W_\pm \varphi(H_0), W^* \varphi(H) \) leave \( \mathcal{D}(D) \) invariant.

**Proof.** Consider first \([D, W_+ \varphi(H_0)]\). Given \( \varphi \in C_0^\infty((0, \infty) \setminus \sigma_p(H)) \) let \( \psi \in C_0^\infty((0, \infty) \setminus \sigma_p(H)) \) be identically one on the support of \( \varphi \). The following computation is justified using the mollified generator of dilation, \( D(\lambda) = i\Lambda D(D+i\lambda)^{-1} \), see [4, 7]. This step is omitted here and in the sequel. One finds as bounded operators from \( H^{2,0} \) to \( H^{-2,0} \):

\[
[D, \psi(H)e^{itH} \varphi(H)e^{-itH} \psi(H_0)]
\]

\[
= \psi(H)[D, e^{itH} \varphi(H)e^{-itH_0}] \psi(H_0) + \psi(H)[D, e^{itH} \varphi(H)e^{-itH_0}] \psi(H_0)
\]

\[
+ \psi(H)[D, \psi(H)e^{itH} \varphi(H)e^{-itH_0}] \psi(H_0) + \psi(H)[D, \psi(H)e^{itH} \varphi(H)e^{-itH_0}] \psi(H_0) .
\]

The last two terms above are bounded operators on \( \mathcal{H} \) (Lemma 2.4 (ii)), uniformly bounded in \( t \).

In the following computation one uses

\[
[-iH_0, D]^a = -2H_0
\]

\[
[-iH, D]^a = -2H_0 - i[V, D]^a = -2H + 2V - i[V, D]^a
\]

valid as bounded operators from \( H^{2,0} \) to \( H^{-2,0} \).
\[ \psi(H)[D, e^{itH} \varphi(H)e^{-itH_0}]\psi(H_0) \]

\[ = \psi(H)e^{itH} \left\{ \int_0^t e^{-isH}[ - iH, D]e^{isH} ds \varphi(H) + [D, \varphi(H)] - \right. \]

\[ - \varphi(H) \int_0^t \int_0^t \left. \right\} e^{-itH_0}[ - iH_0, D]e^{itH_0} ds \right\} e^{-itH_0}\psi(H_0) \]

\[ = \psi(H)e^{itH} \left\{ \int_0^t e^{-isH} \left( - 2H + 2V - i[V, D] \right)e^{isH} ds \varphi(H) + \right. \]

\[ + [D, \varphi(H)] + 2t \varphi(H)H_0 \right\} e^{-itH_0}\psi(H_0) \]

\[ = - 2t \varphi(H)e^{itH}Ve^{-itH_0}\psi(H_0) + \]

\[ + \psi(H)e^{itH}[D, \varphi(H)]e^{-itH_0}\psi(H_0) + \]

\[ + \psi(H)e^{itH_2} \int_0^t e^{-isH}Ve^{isH} ds \varphi(H)e^{-itH_0}\psi(H_0). \]

(2.5)

The last two terms define bounded operators on \( \mathcal{H} \), with norm uniformly bounded in \( t \). The first term is treated as follows. Let \( f, g \in \mathcal{D}(D) \) be given. Using the local \( H \)- and \( H_0 \)-smoothness properties of \( V \) (cf. the proof of Lemma 2.3) one can find a sequence \( t_n \to \infty \) as \( n \to \infty \), such that

\[ \lim_{n \to \infty} t_n \langle f, \varphi(H)e^{itH}Ve^{-itH_0}\psi(H_0)g \rangle = 0. \]

Since the remaining terms in (2.4) and (2.5) are bounded operators on \( \mathcal{H} \), uniformly bounded in \( t \), one finds, using the intertwining relation and \( \varphi(H_0) = \psi(H_0)\varphi(H_0)\psi(H_0) \)

\[ |\langle Df, W_+ \varphi(H_0)g \rangle - \langle f, W_+ \varphi(H_0)Dg \rangle| \]

\[ = \left| \lim_{n \to \infty} \langle f, [D, \psi(H)e^{itH} \varphi(H)e^{-itH_0}\psi(H_0)]g \rangle \right| \]

\[ \leq C \| f \| \cdot \| g \| \]

with \( c > 0 \) independent of \( f \) and \( g \). This proves the result for \( W_+ \varphi(H_0) \). A similar proof holds for \( W_- \varphi(H_0) \). Since the wave operators are asymptotically complete, \( W_+ \varphi(H) = s - \lim_{t \to - \infty} e^{itH_0}e^{-itH} \varphi(H) \), and an analogous proof can be given.

**Proof of Theorem 2.2.** It suffices to prove (2.2) for a dense subset of \( E_0([a, b])\mathcal{H} \), since both sides in (2.2) define bounded quadratic forms on this space. Let \( f, g \in E_0([a, b])\mathcal{H} \) with \( Ff, Fg \) smooth with compact support in
(0, \infty) \setminus \sigma_p(H)$, and in particular $f, g \in \mathcal{D}(D)$. As noted above [1; Theorem 3.8] remains valid under Assumption 2.1. Thus one has

$$
\langle f, TH_0g \rangle = -\frac{1}{2} \langle f, S^*[D, S]g \rangle = -\frac{1}{2} (\langle Sf, DSg \rangle - \langle f, Dg \rangle).
$$

A computation as quadratic form on $\mathcal{D}(D) \cap \mathcal{D}(H_0)$ yields

$$
\frac{d}{dt} (e^{itH}e^{-itH_0}De^{itH_0}e^{-itH})
$$

(2.6)

$$
= -2e^{itH} \tilde{V} e^{-itH} + 2 \frac{d}{dt} (te^{itH}Ve^{-itH}).
$$

Write $W(t) = e^{itH}e^{-itH_0}$. Let $u = \varphi(H) \tilde{u}$, $v = \varphi(H) \tilde{v}$, $\tilde{u}, \tilde{v} \in \mathcal{D}(D)$, $\varphi \in C_0^\infty((0, \infty) \setminus \sigma_p(H))$. Integrating (2.6) gives

$$
\langle W(t)*u, DW(t)*v \rangle = \langle u, Dv \rangle - 2 \int_0^t \langle u, e^{isH} \tilde{V} e^{-isH}v \rangle ds +
$$

$$
+ 2t \langle u, e^{itH}V e^{-itH}v \rangle.
$$

The local $H$-smoothness of $V$ implies the existence of a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$
(2.7) \quad \lim_{n \to \infty} t_n \langle u, e^{it_nH} \tilde{V} e^{-it_nH}v \rangle = 0
$$

(see also Remark 2.8 (i)).

Lemma 2.3 now implies

$$
\lim_{n \to \infty} \langle W(t_n)*u, DW(t_n)*v \rangle = \langle u, Dv \rangle - 2 \int_0^\infty \langle u, e^{isH} \tilde{V} e^{-isH}v \rangle ds.
$$

To conclude that the left hand side equals $\langle W*_u, DW*_v \rangle$ it suffices to show that $\|DW(t)*v\| \leq c$ for all $t \in \mathbb{R}$. To prove this estimate let $\psi \in C_0^\infty((0, \infty) \setminus \sigma_p(H))$ be identically one on the support of $\varphi$. Write $v = \varphi(H)(D + i)^{-1}v_1$.

$$
\|DW(t)*v\| = \|De^{itH_0}\psi(H)e^{-itH}\varphi(H)(D + i)^{-1}v_1\|
$$

$$
\leq \|D\varphi(H)(D + i)^{-1}v_1\| + \|[D, e^{itH_0}\psi(H)e^{-itH}]\varphi(H)(D + i)^{-1}v_1\|.
$$

As in (2.5) above one finds in $\mathscr{B}(\mathcal{H})$
\[ [D, e^{itH_0} \psi(H)e^{-itH}] \varphi(H) = e^{itH_0}[[D, \psi(H)] + 2tV\psi(H) - 2\psi(H) \int_0^t e^{-i\sigma H} \tilde{V} e^{i\sigma H} ds] \varphi(H)e^{-itH}. \]

(2.8)

Lemma 2.5 and Assumption 2.1 imply
\[ \|Ve^{-itH} \varphi(H)(D+i)^{-1} v\| \leq c(1+|t|)^{-1}. \]

The estimate \( \|DW(t) v\| \leq c \) now follows from Lemma 2.3.

Thus one has
\[ \langle W^*_u, DW^*_v \rangle = \langle u, Dv \rangle - 2 \int_0^\infty \langle u, e^{isH} \tilde{V} e^{-isH} v \rangle ds \]
for \( u = \varphi(H) \tilde{u}, v = \varphi(H) \tilde{v}, \tilde{u}, \tilde{v} \in \mathcal{D}(D) \). Similarly, one finds
\[ \langle W^*_u, DW^*_v \rangle = \langle u, Dv \rangle + 2 \int_{-\infty}^0 \langle u, e^{isH} \tilde{V} e^{-isH} v \rangle ds \]
and thus
\[ \langle W^*_u, DW^*_v \rangle - \langle W^*_u, DW^*_v \rangle = -2 \int_{-\infty}^\infty \langle u, e^{isH} \tilde{V} e^{-isH} v \rangle ds. \]

Take now \( u = W_- \varphi(H_0) f, v = W_- \varphi(H_0) g, f, g \in \mathcal{D}(D) \). Then \( W^*_u = \varphi(H_0) f, W^*_u = S\varphi(H_0) f \), etc. and the equation (2.2) has been proved for the dense set
\[ \{ \varphi(H_0) f \mid f \in \mathcal{D}(D), \varphi \in C_0^\infty((a, b)) \}. \]

**Remark 2.8** (i) Note that (2.7) can be improved, since only a dense set of \( u, v \) is considered. Lemma 2.5 and interpolation imply
\[ \|(1+x^2)^{-\delta/2} e^{-itH} \varphi(H)(D+i)^{-1} \| \leq c(1+|t|)^{-\delta}, \quad 0 \leq \delta \leq 1. \]

Under assumption 2.1, \( V(x) = O(|x|^{-1-\epsilon}) \) as \( |x| \to \infty \), so one has
\[ \|u, e^{itH} V e^{-itH} v\| \leq \|(1+x^2)^{-\epsilon/2} e^{-itH} u\| \cdot \|(1+x^2)^{\epsilon/2} e^{-itH} v\| \leq c(1+|t|)^{-1-\epsilon} \]
for \( u = \varphi(H) \tilde{u}, v = \varphi(H) \tilde{v}, \tilde{u}, \tilde{v} \in \mathcal{D}(D) \).

(ii) The computation (2.8) gives a simpler proof of the fact that \( W^*_u \varphi(H) \) leaves \( \mathcal{D}(D) \) invariant, but the result in Lemma 2.7 is stronger. Note that one
has \( \|D\psi(t)\varphi(H)(D+i)^{-1}\| \leq c \) for all \( t \in \mathbb{R} \), but only \( \|D\psi(H)W(t)\varphi(H_0)(D+i)^{-1}\| \leq c \) for all \( t \in \mathbb{R} \), cf. (2.5).

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