MATH. SCAND. 40 (1977), 5–19

ISOMORPHISMS BETWEEN COMPLEXES WITH APPLICATIONS TO THE HOMOLOGICAL THEORY OF MODULES

HANS-BJØRN FOXBY

In a category of modules over a ring the tensor product functor and the homomorphism functor are connected via a number of functorial morphisms. The corresponding morphisms between the derived functors of these functors are isomorphisms in suitable subcategories of the derived category of the category of modules. The aim of this paper is to show some applications of these isomorphisms to some problems of homological nature in the theory of modules over Noetherian rings.

One application is to the theory of finitistic dimensions of Noetherian rings. Recall the following result (due to Gruson and Raynaud [7] and Jensen [12]): If a module over a d-dimensional Noetherian commutative ring is of finite flat dimension (=Tor-dimension), then its projective dimension is at most d. The proof given here is based on a study of the vanishing of the cohomology of the complex RHom (X, Y) where X and Y are bounded complexes of flat modules. This proof seems to be of a completely different nature from that of the original proof.

Another application is to Poincaré series. For a finitely generated module M over a local ring A with residue field k consider the two power series $P^M(t) = \sum_l \beta_l(M)t^l$ and $I^M(t) = \sum_l \mu^l(M)t^l$ where $\beta_l(M)$ and $\mu^l(M)$ are the dimensions of the k-vectorspaces $\operatorname{Tor}_l(k, M)$ and $\operatorname{Ext}^l(k, M)$, respectively. It is proved that these power series are connected via a number of formulas. One of them is

$$I^M(t) = I^A(t)P^M(t^{-1})$$

which holds for a finitely generated module M of finite projective dimension. The main result of the last part of the paper follows easily from this formula, and states that the ring A is a Gorenstein ring, if there exists a non-zero finitely generated module of both finite injective and projective dimension.

The paper is closed by a result that shows the connection between the Intersection conjecture (of Peskine's and Szpiro's) and the notation of infimum of complex used in the first part of the paper.

I take this opportunity to thank Anders Thorup for his stimulation and helpful discussions concerning this material.

Received May 30, 1976.

1. The isomorphisms.

In this first section of the paper we recall some of the properties of the functors ${}^{\mathrm{L}}\otimes$ and RHom in the derived category of the category of modules over the ring A. Here, and in all that follows, the ring A is assumed to be Noetherian, commutative and with a multiplicative identity. Hartshorne's notes [8] serve as a basic reference in the discussions that follow. We shall study complexes of A-modules:

$$\dots \to X^{l-1} \to X^{l} \to X^{l+1} \to \dots$$

here simply denoted by X. Let K denote the triangulated category whose objects are complexes of A-modules and whose morphisms are homotopy equivalence classes of A-homomorphisms of complexes (the A-homomorphisms of complexes are supposed to commute with the differentials of the complexes), cf. [8, p. 25]. Let D be the derived category of the category of A-modules, that is, the localization of K with respect to the multiplicative system consisting of all quasi-isomorphisms of complexes (cf. [8, p. 37]). Let D^{+}, D^{-}, and D^{b} denote the full subcategories of D whose objects are complexes bounded below, bounded above, and bounded on both sides, respectively. Finally the full subcategory of D (respectively of D^{+}, D^{-}, or D^{b}) consisting of complexes with finitely generated cohomology modules is denoted by D_{fg} (and respectively by D_{fg}^{+}, D_{fg}^{-}, $D_{\mathrm{fg}}^{\mathrm{b}}$).

The category of A-modules will be considered as a subcategory of D by viewing a module M as a complex X with $X^{0}=M$ and $X^{l}=0$ for $l\neq 0$. This is a full subcategory of D (cf. [8, p. 40]).

The functor Hom: $K^{\mathrm{op}} \times K \to K$ has a right derived functor RHom: $(D^{-})^{\mathrm{op}} \times D^{+} \to D$. That is, if X is in D^{-} and Y is in D^{+}, and if either X is a complex of projective modules or if Y is a complex of injective modules, then $\mathrm{RHom}\,(X, Y) \cong \mathrm{Hom}\,(X, Y)$ (as complexes in D). In this situation we have $H^{l}(\mathrm{RHom}\,(X, Y)) \cong \mathrm{Mor}_{K}\,(X, Y[l])$ where $Y[l]$ is "the complex Y shifted l steps to the left", that is $Y[l]^{i} = Y^{l+i}$. For details consult [8, Chapter I § 6].

The left derived functor of the functor $\otimes: K \times K \to K$ will be denoted ${}^{\mathrm{L}}\otimes: D^{-} \times D^{-} \to D$. If X and Y are complexes in D^{-} and if one of them consists only of flat modules, then $X\,{}^{\mathrm{L}}\otimes\, Y \cong X \otimes Y$ (in D).

The four fundamental isomorphisms are collected in the following result.

PROPOSITION 1.1. *Let X, Y, and Z be complexes. Then there exist four functorial isomorphisms*:

(1) $X\,{}^{\mathrm{L}}\otimes(Y\,{}^{\mathrm{L}}\otimes Z) \cong (X\,{}^{\mathrm{L}}\otimes Y)\,{}^{\mathrm{L}}\otimes Z$ *for X, Y, and Z all in D^{-}.*

(2) $\mathrm{RHom}\,(X, \mathrm{RHom}\,(Y, Z)) \cong \mathrm{RHom}\,(X\,{}^{\mathrm{L}}\otimes Y, Z)$ *for X and Y in D^{-} and Z in D^{+}.*

(3) $X \overset{L}{\otimes} \mathrm{RHom}(Y,Z) \cong \mathrm{RHom}(\mathrm{RHom}(X,Y),Z)$ *for* X *in* D_{fg}^- *and* Y *and* Z *in* D^b *provided* Z *is isomorphic (in* D*) to a bounded complex of injective modules.*

(4) $\mathrm{RHom}(X,Y) \overset{L}{\otimes} Z \cong \mathrm{RHom}(X, Y \overset{L}{\otimes} Z)$ *for* X *in* D_{fg}^- *and* Y *and* Z *in* D^b *provided* Z *is isomorphic (in* D*) to a bounded complex of flat modules.*

PROOF. (1) and (4) can be found in [8, p. 112].

(2) follows directly from the functorial isomorphism of complexes:

$$\mathrm{Hom}(X, \mathrm{Hom}(Y,Z)) \cong \mathrm{Hom}(X \otimes Y, Z)$$

for complexes X, Y, and Z.

(3) Since there is a functorial homomorphism of complexes:

$$X \otimes \mathrm{Hom}(X,Z) \to \mathrm{Hom}(\mathrm{Hom}(X,Y),Z)$$

for X in D^- and Y and Z in D^b, there exists a functorial morphism: $X \overset{L}{\otimes} \mathrm{RHom}(Y,Z) \to \mathrm{RHom}(\mathrm{RHom}(X,Y),Z)$. This is an isomorphism if $X=A$ and $Y,Z \in D^b$. The functors $- \overset{L}{\otimes} \mathrm{RHom}(Y,Z)$ and $\mathrm{RHom}(\mathrm{RHom}(-,Y),Z)$ are both way-out left, the latter because Z is isomorphic to a bounded complex of injective modules, see [8, Chapter I, Proposition 7.6, p. 80]. Now by a version of the lemma on way-out functors [8, p. 69] we get the desired isomorphism.

Now we recall some facts about dualizing complexes. In order to facilitate the discussion that follows, we assume that the ring A is a homomorphic image of a Gorenstein ring of finite Krull-dimension, but everything that we will state here holds for general dualizing complexes as well, cf. [8]. In fact there is a question to be asked: Is each (local) ring that admits a dualizing complex a homomorphic image of some Gorenstein (local) ring? If the local ring A is Cohen–Macaulay and if A has a dualizing complex, then this dualizing complex is isomorphic to a module, say Ω, and it turns out that any commutative extension of A by Ω is a Gorenstein ring. This has been proved by Fossum in [4].

In the rest of this section of the paper we assume that A is a homomorphic image of a Gorenstein ring of finite Krull-dimension, say $A = R/\mathfrak{r}$, where R is an n-dimensional Gorenstein ring. Let Q be the minimal injective resolution of R (as an R-module) and write $I_{A/R}$ (or just I) for $\mathrm{Hom}_R(A,Q)$. Since $Q^l = \coprod E_R(R/\mathfrak{q})$ where the sum is taken over all prime ideals $\mathfrak{q}$ in R of height l (see Bass [3]) we have

$$I^l = \coprod_{d(\mathfrak{p})=l} E_A(A/\mathfrak{p}),$$

where, for a prime ideal $\mathfrak{p}$, the notation $d(\mathfrak{p})$ denotes the height of the (unique) prime ideal $\mathfrak{q}$ in R with $\mathfrak{q}/\mathfrak{r} = \mathfrak{p}$. Therefore I is a bounded complex of injective

A-modules and each cohomology module $H^l(I) = \mathrm{Ext}_R^l(A, R)$ is finitely generated (so $I \in D_{\mathrm{fg}}^{\mathrm{b}}$). Since the canonical morphism $A \to \mathrm{RHom}(I, I)$ is an isomorphism, the functor $\mathrm{RHom}(\mathrm{RHom}(-, I), I)$ is isomorphic to the identity functor on D_{fg}^- (by Proposition 1.1. (3)) and I is said to be a *dualizing complex*, see p. 258 of [8].

The two natural isomorphisms in the next result follow from Proposition 1.1. (3) and (4).

COROLLARY 1.2. *There exist two functorial isomorphisms involving the dualizing complex I.*

(a) *If X is isomorphic to a bounded complex of injective modules, then* $X \cong I \overset{\mathrm{L}}{\otimes} \mathrm{RHom}(I, X)$.

(b) *If X is isomorphic to a bounded complex of flat modules, then* $X \cong \mathrm{RHom}(I, I \overset{\mathrm{L}}{\otimes} X)$.

Note finally that for each prime ideal $\mathfrak{p}$ in A the localized complex $I_\mathfrak{p}$ is a dualizing complex for the local ring $A_\mathfrak{p}$. If $\mathfrak{q}$ is the prime ideal in R with $\mathfrak{q}/\mathfrak{r} = \mathfrak{p}$ then $I_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{q}}(A_\mathfrak{p}, Q_\mathfrak{q})$.

2. The infimum and the supremum of a complex.

In this section we study what we call the infimum and the supremum of a complex, denoted and defined by, respectively:

$$i(X) = \inf\{l \mid H^l(X) \neq 0\}$$

and

$$s(X) = \sup\{l \mid H^l(X) \neq 0\}$$

with the usual conventions that if X is a trivial complex (that is, X is acyclic, or in other words X is isomorphic to the zero complex) then $i(X) = \infty$ and $s(X) = -\infty$. Note that if $i(X) > -\infty$ then X is isomorphic to a complex, say J, of injective modules such that $J^l = 0$ for $l < i(X)$. Similarly if $s(X) < \infty$ then X is isomorphic to a complex, say F, of free (or projective, or just flat) modules such that $F^l = 0$ for $l > s(X)$. Using these facts it is easy to prove the following simple but useful lemma.

LEMMA 2.1. *For the non-trivial complexes X and Y we have the following two inequalities:*

(1) *If X is in D^- and Y is in D^+, then*

$$i(RHom(X, Y)) \geqq -s(X) + i(Y).$$

Furthermore, if $s = s(X)$ and $i = i(Y)$, then

$$H^{-s+i}(\mathrm{RHom}\,(X, Y)) = \mathrm{Hom}\,(H^s(X), H^i(Y)) .$$

(2) *If X and Y are in D^-, then*

$$s(X \overset{\mathrm{L}}{\otimes} Y) \leqq s(X)+s(Y) .$$

Furthermore, if $s=s(X)$ and $t=s(Y)$, then

$$H^{s+t}(X \overset{\mathrm{L}}{\otimes} Y) = H^s(X) \otimes H^t(Y) .$$

Proof. (1). We may assume that Y is a complex of injective modules and that $X^l=0$ for $l>s$ and $Y^l=0$ for $l<i$. Then $H^{-s+i}(\mathrm{RHom}\,(X, Y)) = \mathrm{Mor}_K\,(X[s], Y[i])$ and this module is isomorphic to $\mathrm{Hom}\,(H^s(X), H^i(Y))$.
The proof of (2) is even easier.

Proposition 2.2. *Let X in D^b_{fg} and Y in D^+ be non-trivial complexes, and assume that* $\mathrm{Supp}\, H^i(Y) \subseteqq \bigcup_l \mathrm{Supp}\, H^l(X)$ *where $i=i(Y)$. Then*

$$i(\mathrm{RHom}\,(X, Y)) \leqq -i(X)+i(Y) .$$

Proof. Choose $\mathfrak{p}$ in $\mathrm{Ass}\, H^i(Y)$ and let B denote the local ring $A_\mathfrak{p}$. By hypothesis $X_\mathfrak{p}$ is a non-trivial complex, so $s(X_\mathfrak{p}) \geqq i(X_\mathfrak{p}) \geqq i(X)$. Since $X \in D^-_{\mathrm{fg}}$ we have an isomorphism:

$$\mathrm{RHom}_A\,(X, Y)_\mathfrak{p} = \mathrm{RHom}_B\,(X_\mathfrak{p}, Y_\mathfrak{p})$$

(both sides are contra-variant way-out right functors in X), and hence $i(\mathrm{RHom}_B\,(X_\mathfrak{p}, Y_\mathfrak{p})) \geqq i(\mathrm{RHom}\,(X, Y))$. For $s=s(X_\mathfrak{p})$ the B-module $H^s(X_\mathfrak{p})$ (which is $H^s(X)_\mathfrak{p}$) is finitely generated. Hence we have $\mathrm{Hom}_B\,(H^s(X_\mathfrak{p}), H^i(Y_\mathfrak{p})) \neq 0$ since $\mathfrak{p}B$ is in $\mathrm{Ass}_B\, H^i(Y_\mathfrak{p})$. This gives $i(\mathrm{RHom}_B\,(X_\mathfrak{p}, Y_\mathfrak{p})) = -s+i$, by part (1) of the Lemma, and we are done.

Corollary 2.3. *Assume that A is a homomorphic image of a Gorenstein ring of finite Krull-dimension (so A admits a dualizing complex I). Let the complex X be isomorphic to a bounded complex of flat modules. Then*

$$i(I \overset{\mathrm{L}}{\otimes} X) \geqq i(I)+i(X) .$$

Proof. From Corollary 1.2.(b) and the above Proposition we get $i(X) = i(\mathrm{RHom}\,(I, I \overset{\mathrm{L}}{\otimes} X)) \leqq -i(I)+i(I \overset{\mathrm{L}}{\otimes} X)$.

The following technical result turns also out to be useful.

Lemma 2.4. *Suppose the complex X is in D^b and the complex Y is in D^+. Then*

$$s(\mathrm{RHom}\,(X, Y)) \leqq \sup_l s(\mathrm{RHom}\,(H^l(X), Y[l]))$$

Proof. For any number l let $X^{\geqq l}$ denote the truncated complex

$$\ldots \to 0 \to 0 \to (X^l/\mathrm{Im}\,(X^{l-1} \to X^l)) \to X^{l+1} \to X^{l+2} \to \ldots$$

Note that we have a triangle in D

$$\begin{tikzcd} & X^{\geqq l+1} \arrow[dl, squiggly] & \\ H^l(X)[-l] \arrow[rr] & & X^{\geqq l} \arrow[ul] \end{tikzcd}$$

(cf. [8, p. 70]).

Suppose that the supremum on the right hand side is at most t. Since $X \cong X^{\geqq l}$ for all l small enough we are required to prove that $s(\mathrm{RHom}\,(X^{\geqq l}, Y)) \leqq t$ for all l. This will be done by descending induction on l. For $l > s(X)$ $X^{\geqq l}$ is trivial, so assume that $s(\mathrm{RHom}\,(X^{\geqq l+1}, Y)) \leqq t$. From the above triangle we obtain another one

$$\begin{tikzcd} & \mathrm{RHom}\,(H^l(X), Y[l]) \arrow[dl, squiggly] & \\ \mathrm{RHom}\,(X^{\geqq l+1}, Y) \arrow[rr] & & \mathrm{RHom}\,(X^{\geqq l}, Y) \arrow[ul] \end{tikzcd}$$

and from this triangle we get $s(\mathrm{RHom}\,(X^{\geqq l}, Y)) \leqq t$ as desired.

Remark 2.5. From [3] it follows that $i(I) = \mathrm{ht}_R\, \mathfrak{r}$ when $A = R/\mathfrak{r}$ where R is Gorenstein. Note also $i(I_\mathfrak{p}) = d(\mathfrak{p}) - \mathrm{ht}_A\, \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ in A.

Now we are ready for the main result of this section.

Theorem 2.6. *Assume that A is a homomorphic image of a Gorenstein ring of finite Krull-dimension. Let both X and Y be isomorphic to bounded complexes of flat modules. Then*

$$s(\mathrm{RHom}\,(X, Y)) \leqq -i(X) + s(Y) + \dim A$$

Proof. The notation will be as in the end of section 1. In particular $n = \dim R$. Let $i = i(X)$, $s = s(Y)$, and $d = \dim A$.

Note first of all that $\mathrm{RHom}\,(X, Y) \cong \mathrm{RHom}\,(X \overset{\mathrm{L}}{\otimes} I, I \overset{\mathrm{L}}{\otimes} Y)$ by Corollary 1.2.(b) and Proposition 1.1.(2). Then choose Z in D^+ isomorphic to $X \overset{\mathrm{L}}{\otimes} I$ such that $Z^l = 0$ for $l < i + i(I)$ (this is possible by Corollary 2.3). Choose also a complex of flat modules $P \in D^{\mathrm{b}}$ isomorphic to Y such that $P^l = 0$ for $l > s$. Here $I \otimes P$ is a complex of injective modules, $I \otimes P \cong I \overset{\mathrm{L}}{\otimes} Y$, and $(I \otimes P)^l = 0$ for $l > s + n$. Whence we get $\mathrm{Hom}\,(Z, I \otimes P)^l = 0$ for $l > -i + s + (n - i(I))$. Therefore $s(\mathrm{RHom}\,(X, Y)) = s(\mathrm{Hom}\,(Z, I \otimes P)) \leqq -i + s + d$ as desired in the (important) case where $d=n-i(I)$ (e.g. if A is local, cf. the Remark preceding the Theorem).

The proof in the general case is more complicated. Let e be the smallest of the numbers $s-n$ and $\inf\{l \mid P^l \neq 0\}$, and let F be a complex of free modules such that $F \cong Y$ and $F^l=0$ for $l>s$. Since $i(M\otimes F)=i(M\otimes P)\geqq e$ for all modules M, we have

$$\mathrm{Tor}_l(M,N) = 0$$

for $l>0$ and $N=F^e/\mathrm{Im}\,(F^{e-1} \to F^e)$, so $F^{\geqq e}$ is a complex of flat modules (one flat module and $s-e$ free modules). Since $J=I\otimes(F^{\geqq e})\cong I\overset{\mathrm{L}}{\otimes} Y$, we have

$$\mathrm{RHom}\,(X,Y) \cong \mathrm{Hom}\,(Z,J)\,.$$

Now assume $s(\mathrm{RHom}\,(X,Y))>-i+s+d$ and we seek a contradiction. By Lemma 2.4 there exist numbers p and l, with $l>-i+s+d$, such that $\mathrm{Hom}\,(H^p(Z),J^{p+l})\neq 0$. So it is possible to pick a prime ideal $\mathfrak{p}$ in both $\mathrm{Supp}\,H^p(Z)$ and $\mathrm{Ass}\,J^{p+l}$. Also

$$p \geqq i(Z_{\mathfrak{p}}) \geqq i(I_{\mathfrak{p}})+i(X_{\mathfrak{p}}) \geqq d(\mathfrak{p})-\mathrm{ht}\,\mathfrak{p}+i$$

by Corollary 2.3 and Remark 2.5. Hence $d(\mathfrak{p})\leqq p-i+d$.

From this it follows that

$$e \leqq s-n \leqq d(\mathfrak{p})+s-n \leqq p-i+s+d-n < p+l-n$$

and so

$$J^{p+l} = \coprod_{q=p+l-s}^{n} I^q\otimes F^{p+l-q}\,.$$

Since $\mathfrak{p}\in\mathrm{Ass}\,J^{p+l}$ we get $\mathfrak{p}\in\mathrm{Ass}\,(I^q\otimes F^{p+l-q})$ for some $q\geqq p+l-s$ and hence $\mathfrak{p}\in\mathrm{Ass}\,I^q$, that is $d(\mathfrak{p})=q\geqq p+l-s>p-i+d$. This contradicts the fact that $d(\mathfrak{p})\leqq p-i+d$ as proved above, and we are done with the proof of the Theorem.

3. Flat dimension and projective dimension.

In this section we will apply the preceding Theorem to the homological dimension theory of a module, but let us first fix the notation.

For an A-module M set

$\mathrm{id}_A\,M=$ the injective dimension of M,
$\mathrm{pd}_A\,M=$ the projective (or homological) dimension of M, and
$\mathrm{fd}_A\,M=$ the flat (or weak homological) dimension of M.

The corresponding finitistic dimensions of the ring will be denoted as follows.

FID (A) = the finitistic injective dimension,
FPD (A) = the finitistic projective dimension, and
FFD (A) = the finitistic flat dimension

(and each of these is the supremum of respectively $\mathrm{id}_A M$, $\mathrm{pd}_A M$, and $\mathrm{fd}_A M$, when M in each case runs through the A-modules of finite injective, projective and flat dimensions, respectively).

These finitistic dimensions have been studied by Auslander and Buchsbaum, see [1].

THEOREM 3.1. (Auslander and Buchsbaum) *The following equalities hold:*

$$\mathrm{FID}(A) = \mathrm{FFD}(A) = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{depth} A_{\mathfrak{p}} .$$

In particular, $\dim A - 1 \leqq FFD(A) \leqq \dim A$, *and for A local* $\mathrm{FFD}(A) = \dim A$ *if and only if A is Cohen–Macaulay.*

In 1962 Bass constructed for each integer $d \leqq \dim A$ an A-module M with $\mathrm{pd}_A M = d$, see [2], and hence $\mathrm{FPD}(A) \geqq \dim A$. It is easy to see that $\mathrm{FPD}(A) = \dim A$ if A is Gorenstein. In fact, Bass stated that he knew of no ring with strict inequality. The problem whether such a ring could exist was solved ten years later by Gruson and Raynaud (see [7]).

THEOREM 3.2. (Bass, Gruson and Raynaud) *Let A be a ring, then*

$$\mathrm{FPD}(A) = \dim A .$$

In the following result (see Jensen [12]) the ring R need not be Noetherian (nor commutative).

PROPOSITION 3.3. (Jensen). *Let N be a flat (left) R-module and suppose (left-)* $\mathrm{FPD}(R) < \infty$. *Then* $\mathrm{pd}_R M < \infty$.

As an immediate concequence of these two results we obtain the following corollary.

COROLLARY 3.4. *If* $\mathrm{fd}_A M < \infty$, *then* $\mathrm{pd}_A M \leqq \dim A$.

Note that Theorem 3.2 and Proposition 3.3 (in the (commutative) Noetherian case) follow directly from this Corollary. Also the inequality $\mathrm{FFD}(A) \leqq \dim A$ follows of course from the Corollary.

We will now give an alternative proof of this Corollary in the case where A is a homomorphic image of a Gorenstein ring of finite Krull-dimension, e.g. if A

is an essentially finitely generated algebra over a field, or if A is a complete local ring. In fact, in their proof of Theorem 3.2 Gruson and Raynaud first prove the result for complete local rings and then deduce the general result from this special one.

PROOF OF COROLLARY 3.4. Let M be a module with $\mathrm{fd}_A M < \infty$ and write $d = \dim A$. In the exact sequence

$$0 \to N \to F_d \to \dots \to F_0 \to M \to 0$$

each F_i is assumed to be a free module. Let $K = \mathrm{Coker}(N \to F_d)$. From Theorem 2.6 it follows that $s(\mathrm{RHom}(M,N)) \leqq d$, and hence $\mathrm{Ext}^1(K,N) = \mathrm{Ext}^{d+1}(M,N) = 0$, so K is projective, that is $\mathrm{pd}_A M \leqq d$.

4. Poincaré series.

In this section A is assumed to be local with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. We will give some formulas connecting minimal injective resolutions with minimal free resolutions. We start with an illustrative example.

EXAMPLE. Let M be a finitely generated A-module of finite injective dimension. In the minimal injective resolution of M:

$$0 \to M \to E^0 \to E^1 \to \dots \to E^d \to 0$$

each of the injective modules E^j contains a certain number, say $\mu^i(M)$, of copies of $E_A(k)$. (See Bass [3] where the equalities $d = \mathrm{id}_A M = \mathrm{depth}\, A$ are demonstrated.)

Now assume that A is a Gorenstein ring of dimension d. Then it is well-known that $\mathrm{pd}_A M$ is also finite. In fact, in the minimal free resolution of M:

$$0 \to F_p \to \dots \to F_1 \to F_0 \to M \to 0$$

the free module F_j has rank $\mu^{d-j}(M)$, cf. [5].

For a vector space V over the field k the dimension of V is denoted by $[V:k]$. Let $\mathbf{Z}((t))$ denote the ring of formal Laurent series with integral coefficients (that is formal sums $\sum_{l \geqq d} a_l t^l$ where d and all a_l are integers.)

DEFINITIONS. (1) For X in D^+_{fg} and l in $\mathbf{Z}$ let

$$\mu^l(X) = [H^l(\mathrm{RHom}_A(K,X)):k] \quad \text{and}$$
$$I^X(t) = \sum_l \mu^l(X) t^l \in \mathbf{Z}((t)) .$$

(2) For X in D^-_{fg} and l in $\mathbf{Z}$ let

$$\beta_l(X) = [H^{-l}(k\overset{\mathrm{L}}{\otimes}_A X):k] \quad \text{and}$$
$$P^X(t) = \sum_l \beta_l(X)t^l \in \mathbf{Z}((t)) .$$

REMARK. For a module M this definition of $\mu^l(M)$ coincides with the one in Bass [3], and $\beta_l(M)$ is the lth Betti-number, so $P^M(t)$ is just the Poincaré series for M.

THEOREM 4.1. *Let X be in D_{fg} and Y be in D^+_{fg}.*
(a) *If $X \in D^{\mathrm{b}}_{\mathrm{fg}}$ then*

$$\mu^l(\mathrm{RHom}\,(X, Y)) = \sum_p \beta_p(X)\mu^{l-p}(Y)$$

for all l. And therefore

$$I^{\mathrm{RHom}\,(X,Y)}(t) = P^X(t)I^Y(t) .$$

(b) *If Y is isomorphic to a bounded complex of injective modules, then*

$$\beta_l(\mathrm{RHom}\,(X, Y)) = \sum_p \mu^p(X)\mu^{p-l}(Y)$$

for all l. And therefore

$$P^{\mathrm{RHom}\,(X,Y)}(t) = I^X(t)I^Y(t^{-1}) .$$

THEOREM 4.2. *Let X and Y be in D^-_{fg}.*
(a) *If $X \in D^{\mathrm{b}}_{\mathrm{fg}}$ then*

$$\beta_l(X\overset{\mathrm{L}}{\otimes} Y) = \sum_p \beta_p(X)\beta_{l-p}(Y)$$

for all l, and so

$$P^{X\overset{\mathrm{L}}{\otimes} Y}(t) = P^X(t)P^Y(t) .$$

(b) *If Y is isomorphic to a bounded complex of flat modules, then*

$$\mu^l(X\overset{\mathrm{L}}{\otimes} Y) = \sum_p \mu^p(X)\beta_{p-l}(Y)$$

for all l, and so

$$I^{X\overset{\mathrm{L}}{\otimes} Y}(t) = I^X(t)P^Y(t^{-1}) .$$

Note that all the four summations of integers in these two theorems are finite.

PROOFS. 4.1. (a). If we let $V = \mathrm{RHom}_A(k, \mathrm{RHom}_A(X, Y))$, then

$$V \cong \mathrm{RHom}_A(k^{\mathrm{L}}\otimes_A X, Y)$$

by Proposition 1.1.(2). Now $k^{\mathrm{L}}\otimes_A X \cong (k^{\mathrm{L}}\otimes_A X)\otimes_k k$ (in $D(k)$, the derived category of the category of vector spaces over k) and hence

$$V \cong \mathrm{RHom}_A((k^{\mathrm{L}}\otimes_A X)\otimes_k k, Y) \cong \mathrm{RHom}_k(k^{\mathrm{L}}\otimes_A X, \mathrm{RHom}_A(k, Y))$$

by a version of Proposition 1.1(2).

Each complex Z in $D(k)$ is isomorphic to its cohomology complex $H(Z)$ (since each short-exact sequence of vector spaces splits). Therefore

$$V \cong \mathrm{Hom}_k(H(k^{\mathrm{L}}\otimes_A X), H(\mathrm{RHom}_A(k, Y)) ,$$

and this is a complex with zero differentials. Whence

$$\begin{aligned} H^l(V) &= \mathrm{Hom}_k(H(k^{\mathrm{L}}\otimes_A X), H(\mathrm{RHom}_A(k, Y)))^l \\ &= \prod_p \mathrm{Hom}_k(H^{-p}(k^{\mathrm{L}}\otimes_A X), H^{l-p}(\mathrm{RHom}_A(k, Y))) \end{aligned}$$

and we are done.

The proof of 4.1.(b) is very similar. We get

$$\begin{aligned} k^{\mathrm{L}}\otimes_A \mathrm{RHom}_A(X, Y) &\cong \mathrm{RHom}_A(\mathrm{RHom}_A(k, X), Y) \\ &\cong \mathrm{RHom}_k(\mathrm{RHom}_A(k, X), \mathrm{RHom}_A(k, Y)) , \end{aligned}$$

by Proposition 1.1.

Also 4.2.(a) and (b) follow from Proposition 1.1.

Now we are going to apply these results to modules. If we let $X = A$ and $Y = N$ we get the following Corollary.

COROLLARY 4.3. *Let N be a finitely generated module.*

(1) *If* $\mathrm{id}_A N < \infty$ *then* $P^N(t) = I^A(t)\, I^N(t^{-1})$.
(2) *If* $\mathrm{pd}_A N < \infty$ *then* $I^N(t) = I^A(t)\, P^N(t^{-1})$.

That formulas like these should hold was suggested by Birger Iversen.

It is well-known that the ring A is Gorenstein if and only if the class of finitely generated modules of finite injective dimension coincides with the class of finitely generated modules of finite projective dimension. The next result shows that: if there is a non-zero module in the intersection of these two classes, then A is Gorenstein. Herzog [9] has proved this if the module N in the intersection satisfies $\mathrm{pd}_A N \leqq 3$ and both A and N are Cohen–Macaulay.

Corollary 4.4. *If there exists a finitely generated non-zero module N such that both $\mathrm{id}_A N$ and $\mathrm{pd}_A N$ are finite, then the ring A is a Gorenstein ring.*

Proof. Let $d = \operatorname{depth} A$. Then $d = \mathrm{id}_A N$ and $d \geqq \mathrm{pd}_A N$ (cf. [3]). From Corollary 4.3 we get

$$0 = \beta_l(N) = \mu^l(A)\mu^0(N) + \ldots + \mu^{d+l}(A)\mu^d(N) \qquad \text{for } l > \mathrm{pd}_A N .$$

That is, $\mu^i(A)\mu^{i-l}(N) = 0$ for $l \leqq i \leqq l+d$. In particular $\mu^{l+d}(A)\mu^d(N) = 0$. But $\mu^d(N) \neq 0$ (see Bass [3]), so $\mu^{l+d}(A) = 0$ for all $l > \mathrm{pd}_A N$. Hence we have proved that A is Gorenstein (cf. again Bass [3]).

Corollary 4.5. *Assume that A is a d-dimensional Cohen–Macaulay local ring that is a homomorphic image of a Gorenstein local ring. Let Ω be the dualizing module, and let C, M, and N be finitely generated non-zero modules such that C is Cohen–Macaulay of grade g, $\mathrm{pd}_A M < \infty$, and $\mathrm{id}_A N < \infty$.*

(a) *If $g = 0$ (that is $\operatorname{depth} C = d$), then*

$$\begin{aligned} I^{\mathrm{Hom}(C,N)}(t) &= P^C(t)\, I^N(t)\,, \\ P^{\mathrm{Hom}(C,N)}(t) &= I^C(t)\, I^N(t^{-1})\,, \\ P^{C\otimes M}(t) &= P^C(t)\, P^M(t), \qquad \textit{and} \\ I^{C\otimes M}(t) &= I^C(t)\, P^M(t^{-1})\,. \end{aligned}$$

(b) *For all l*

$$\begin{aligned} \mu^l(\mathrm{Ext}^g(C,\Omega)) &= \beta_{l+g-d}(C) \qquad \textit{and} \\ \beta_l(\mathrm{Ext}^g(C,\Omega)) &= \mu^{l+d-g}(C) \end{aligned}$$

(c) *For all l*

$$\begin{aligned} \beta_l(\mathrm{Hom}(\Omega,N)) &= \mu^{d-l}(N) \qquad \textit{and} \\ \mu^l(\Omega\otimes M) &= \beta_{d-l}(M)\,. \end{aligned}$$

Most of these formulas are known from Herzog and Kunz [10] and [5].

Proof. See e.g. [6, Lemma 3 (τ) and (σ)] and [5, Lemma 3.3].

Corollary 4.6. *For the finitely generated non-zero modules M and N let $s = \operatorname{depth} M$, $d = \operatorname{depth} A$ and assume $\mathrm{id}_A N < \infty$. Then*

$$\beta_0(\mathrm{Ext}^{d-s}(M,N)) = \mu^s(M)\,\mu^d(N)\,.$$

PROOF. We have $s(k{}^{\mathrm{L}}\otimes_A \mathrm{RHom}\,(M,N))=s(\mathrm{RHom}\,(M,N))=d-s$, and hence

$$\beta_{s-d}(\mathrm{RHom}\,(M,N)) = \beta_0(\mathrm{Ext}^{d-s}\,(M,N))$$

by Lemma 2.1. Therefore the desired assertion follows from Theorem 4.1 (6).

COROLLARY 4.7. *For the finitely generated non-zero modules M and N let*

$$g = \inf\{l \mid \mathrm{Ext}^l\,(M,N)\neq 0\}$$

(so g is the maximal length of an N-regular sequence in the annihilator of M). *Then*

$$\mu^0(\mathrm{Ext}^g\,(M,N)) = \beta_0(M)\,\mu^g(N)\,.$$

PROOF. By Lemma 2.1 (1) we have $i(\mathrm{RHom}\,(k,\mathrm{RHom}\,(M,N)))\geqq g$ and $\mu^g(\mathrm{RHom}\,(M,N))=\mu^0(\mathrm{Ext}^g\,(M,N))$. Hence we are done by Theorem 4.1 (a).

We close this section with a corollary that follows directly from Corollary 4.3. (2).

COROLLARY 4.8. *Let N be a finitely generated non-zero module of projective dimension p $(<\infty)$. Then*

$$\mu^{s-p}(N) = \mu^s(A)\,\beta_p(N)$$

where $s=\mathrm{depth}\,A$.

5. Connection to the Intersection conjecture.

For a finitely generated modules M and B over the local ring A the *Intersection conjecture* states:

$$\dim_A B \leqq \mathrm{pd}_A M \quad \textit{if} \quad \dim_A (B\otimes_A M) = 0\,.$$

This conjecture holds when the ring A is of equicharacteristic, see Peskine and Szpiro [14] and Hochster [11], but also for general rings if M satisfies special conditions (e.g. $\mathrm{pd}_A M\leqq 2$ or $\mathrm{grade}_A M\leqq 1$).

In the following we will assume that B is cyclic, say $B=A/\mathfrak{a}$ (and this restriction costs no loss of generality). Write $I_B=\mathrm{RHom}_R\,(B,R)$, that is, I_B denotes the dualizing complex for the ring B.

RESULT. *If* $\mathrm{pd}_A M<\infty$ *and* $\dim_A (B\otimes_A M)=0$ *then*

$$\dim B-\mathrm{pd}_A M = i(I_B{}^{\mathrm{L}}\otimes_A M)-i(I_B)\,.$$

Proof. Since $\mathrm{pd}_A M<\infty$ we have

$$\text{(0)} \qquad \mathrm{RHom}_B(I_B, I_B{}^{\mathrm{L}}\otimes_A M) \cong B^{\mathrm{L}}\otimes_A M$$

by (a version of) Proposition 1.1.(4). Note that $s=s(I_B)=i(I_B)+\dim B-\operatorname{depth} B$ (cf. Remark 2.5) and that

$$\mathrm{Hom}(H^s(I_B), H^l(I_B{}^{\mathrm{L}}\otimes_A M)) \neq 0$$

for $l=i(I_B{}^{\mathrm{L}}\otimes_A M)$, since $\dim H^i(I_B{}^{\mathrm{L}}\otimes_A M)\leqq 0$ for all i. Therefore the infimum of the left hand side of (0) is

$$\text{(L)} \qquad -s+l = i(I_B{}^{\mathrm{L}}\otimes_A M)-i(I_B)-\dim B+\operatorname{depth} B .$$

The infimum of the right hand side of (0) is:

$$\text{(R)} \qquad \begin{aligned} i(B^{\mathrm{L}}\otimes_A M) &= -\sup\{j \mid \mathrm{Tor}_j^A(B,M)\neq 0\} \\ &= -\mathrm{pd}_A M+\operatorname{depth} B . \end{aligned}$$

Here the last equality follows easily by induction on $\operatorname{depth} B$ using that $\dim \mathrm{Tor}_i^A(B,M)\leqq 0$ for all i.

The assertion of the Result now follows by comparing (L) and (R).

Since the Result states that the Intersection conjecture holds if and only if $i(I_B)\geqq i(I_B{}^{\mathrm{L}}\otimes_A M)$ we are lead to the question.

Question. Do we for all complexes X in $D_{\mathrm{fg}}^{\mathrm{b}}$ and all finitely generated non-zero modules M with $\mathrm{pd}_A M<\infty$ have

$$i(X) \geqq i(X^{\mathrm{L}}\otimes_A M)?$$

It should be noted that there is an affirmative answer in the following two very special cases:

1°. $\mathrm{pd}_A M\leqq 2$. The proof in this case uses however, that such a module M is Tor-rigid provided $\mathrm{grade}_A M\geqq 1$, that is, for all finitely generated modules N we have

$$\mathrm{Tor}_1(M,N) = 0 \Rightarrow \mathrm{Tor}_2(M,N) = 0 ,$$

cf. [14, Proposition (1.4)].

2°. $\mathrm{grade}_A M\leqq 1$. Here is used that the so-called Strong Intersection conjecture holds: $\dim B\leqq \mathrm{grade}_A M$ if $\mathrm{pd}_A M<\infty$ and $\dim(B\otimes_A M)=0$. (Namely: If $\mathrm{grade}_A M=1$ and $\mathrm{pd}_A M<\infty$, there exists a non zero divisor a in A such that $\mathrm{Supp}(A/(a))\leqq \mathrm{Supp}\, M$ (cf. MacRae [13]). Now $\dim(B/aB)=0$ gives $\dim B\leqq 1$.)

REFERENCES

1. M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings* II, Trans. Amer. Math. Soc. 88 (1958), 194–206.
2. H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. 102 (1962), 18–29.
3. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–28.
4. R. Fossum, *Commutative extensions by canonical modules are Gorenstein rings*, Proc. Amer. Math. Soc. 40 (1973), 395–400.
5. H.-B. Foxby, *On the* μ^i *in a minimal injective resolution*, Math. Scand. 29 (1971), 175–186.
6. H.-B. Foxby, *Duality homomorphisms for modules over certain Cohen–Macaulay rings*, Math. Z. 132 (1973), 215–226.
7. L. Gruson and M. Raynaud, *Critères de platitude*, Invent. Math. 13 (1971), 1–89.
8. R. Hartshorne, *Residues and duality*, Lecture Notes Math. 20, Springer-Verlag, Berlin · Heidelberg · New York, 1966.
9. J. Herzog, *Komplexe, Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift (unpublished). Universität Regensburg, Regensburg, Germany.
10. J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen–Macaulay-Rings*, Lecture Notes Math. 238, Springer-Verlag, Berlin · Heidelberg · New York, 1971.
11. M. Hochster, *Topics in the theory of modules over commutative rings*, C.B.M.S. Regional Conference Series in Math. 24, Amer. Math. Soc., Providence R.I., 1975.
12. C. U. Jensen, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra 15 (1970), 15–166.
13. R. E. MacRae, *On an application of the Fitting invariants*, J. Algebra 2 (1965), 153–169.
14. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. Inst. Hautes Études Sci. 42 (1973), 323–395 (47–119).

UNIVERSITY OF COPENHAGEN
DENMARK