UNIQUENESS THEOREMS FOR
MEROMORPHIC FUNCTIONS

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Abstract.

If $f$ is a transcendental meromorphic function, $a$ is an extended complex number and $k$ is a positive integer or $\infty$, let

$$E(a, k, f) = \{z \in \mathbb{C} \mid z \text{ is a zero of } f-a \text{ of order } \leq k\}$$

where $\mathbb{C}$ is the complex plane. If $f_1, f_2$ are distinct meromorphic functions and if there exist distinct extended complex numbers $a_1, \ldots, a_m$ such that $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ for $i = 1, 2, \ldots, n$ where each $k_i$ is a positive integer or $\infty$ with $k_1 \geq k_2 \geq \ldots \geq k_m$, then it is shown that

$$\sum_{i=2}^{m} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} \leq 2.$$

Several consequences are deduced which include a theorem of Nevanlinna and the following result:

If the set of simple zeros of $f_1 - a$ coincides with the set of simple zeros of $f_2 - a$ for seven distinct values of $a$ in the extended complex plane, the $f_1 \equiv f_2$.

1.

We denote by $\mathbb{C}$ the set of all finite complex numbers and by $\bar{\mathbb{C}}$ the extended complex plane consisting of all (finite) complex numbers and $\infty$. By a meromorphic function we mean a transcendental meromorphic function in the plane. We use the usual notations of the Nevanlinna theory of meromorphic functions as explained in [1] and [3].

If $f$ is a meromorphic function, then as in [1], we denote by $S(r, f)$ any quantity satisfying

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as \( r \to \infty \), through all values if \( f \) is of finite order and outside a set of finite linear measure if \( f \) is of infinite order.

If \( f \) is a meromorphic function, \( a \in \mathbb{C} \) and \( k \) is a positive integer or \( \infty \), we denote by \( \bar{n}_k(r,a,f) \) the number of distinct zeros of order \( \leq k \) of \( f-a \) in \( |z| \leq r \) (each zero of order \( \leq k \) is counted only once irrespective of its multiplicity). Thus, in particular, \( \bar{n}_1(r,a,f) \) is the number of simple zeros and \( \bar{n}_2(r,a,f) \) the number of distinct simple and double zeros of \( f-a \) in \( |z| \leq r \). Also \( \bar{n}_\infty (r,a,f) = \bar{n}(r,a,f) \). \( \bar{N}_k(r,a,f) \) is defined in terms of \( \bar{n}_k(r,a,f) \) in the obvious way. Clearly

\[
\bar{n}(r,a,f) \leq \frac{1}{k+1}\{k\bar{n}_k(r,a,f) + n(r,a,f)\}
\]

so that

\[
\bar{N}(r,a,f) \leq \frac{1}{k+1}\{k\bar{N}_k(r,a,f) + N(r,a,f)\}.
\]

We also denote by \( E(a,k,f) \) the subset of \( C \) consisting of all zeros of order \( \leq k \) of \( f-a \). That is

\[
E(a,k,f) = \{ z \in \mathbb{C} \mid z \text{ is a zero of } f-a \text{ of order } \leq k \}.
\]

In particular, \( E(a,\infty,f) = \{ z \in \mathbb{C} \mid f(z) = a \} \) and we denote it simply by \( E(a,f) \).

Nevanlinna proved the following theorem [2, page 48 and 1, Theorem 2.6]

**Theorem A.** If \( f_1, f_2 \) are meromorphic functions and if \( E(a,f_1) = E(a,f_2) \) for five distinct values of \( a \) in \( \mathbb{C} \), then \( f_1 \equiv f_2 \).

In this paper we obtain a general result of which Theorem A appears as a particular case.

Let \( f_1, f_2 \) be meromorphic functions. If \( a \in \mathbb{C} \) and \( k \) is a positive integer or \( \infty \), then for \( r > 0 \), we denote by \( n_0^{(k)}(r,a) \) the number of common zeros of order \( \leq k \) of \( f_1-a \) and \( f_2-a \) in \( |z| \leq r \), each zero of order \( \leq k \) being counted only once irrespective of its multiplicity. In particular \( n_0^{(\infty)}(r,a) \) is the number of common zeros of \( f_1-a \) and \( f_2-a \) in \( |z| \leq r \) (all zeros are considered) and we also denote it simply by \( n_0(r,a) \). As usual, we set

\[
N_0^{(k)}(r,a) = \int_0^r \frac{n_0^{(k)}(r,a) - n_0^{(k)}(0,a)}{t} dt + n_0^{(k)}(0,a) \log r.
\]

We also define

\[
\bar{N}_1^{(k)}(r,a) = \bar{N}_k(r,a,f_1) + \bar{N}_k(r,a,f_2) - 2N_0^{(k)}(r,a)
\]

and write \( \bar{N}_{1,2}(r,a) \) for \( \bar{N}_1^{(\infty)}(r,a) \).
Our main result is the following

**Theorem 1.** Let $f_1, f_2$ be distinct meromorphic functions (that is, $f_1 \not\equiv f_2$). If there exist distinct elements $a_1, \ldots, a_m$ in $\mathbb{C}$ such that $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ for $i = 1, 2, \ldots, m$ for some $k_1, \ldots, k_m$ each of which is a positive integer or $\infty$ with $k_1 \geq k_2 \geq \ldots \geq k_m$, then

$$
\sum_{i=2}^{m} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} \leq 2.
$$

**Proof.** Suppose, first, that $a_1, \ldots, a_m$ are all finite.

We have, by Nevanlinna’s second fundamental theorem, for $j = 1, 2$,

$$(m - 2)T(r, f_j) \leq \sum_{i=1}^{m} \bar{N}(r, a_i, f_j) + S(r, f_j)
\leq \sum_{i=1}^{m} \frac{1}{k_i + 1} \{k_i \bar{N}_{k_i}(r, a_i, f_j) + N(r, a_i, f_j)\} + S(r, f_j),$$

by (1)

$$\leq \sum_{i=1}^{m} \frac{k_i}{k_i + 1} \bar{N}_{k_i}(r, a_i, f_j) + \left( \sum_{i=1}^{m} \frac{1}{k_i + 1} \right) T(r, f_j) + S(r, f_j).$$

So,

$$\left\{ m - 2 - \sum_{i=1}^{m} \frac{1}{k_i + 1} \right\} T(r, f_j) \leq \sum_{i=1}^{m} \frac{k_i}{k_i + 1} \bar{N}_{k_i}(r, a_i, f_j) + S(r, f_j).$$

Adding the two inequalities corresponding to $j = 1$ and $j = 2$, we obtain

$$\left\{ \sum_{i=1}^{m} \frac{k_i}{k_i + 1} - 2 \right\} \left\{ T(r, f_1) + T(r, f_2) \right\}
\leq \sum_{i=1}^{m} \frac{k_i}{k_i + 1} \{ \bar{N}_{k_i}(r, a_i, f_1) + \bar{N}_{k_i}(r, a_i, f_2) \} + S(r, f_1) + S(r, f_2)
= 2 \sum_{i=1}^{m} \frac{k_i}{k_i + 1} N_0^{(k_i)}(r, a_i) + S(r, f_1) + S(r, f_2),$$

since, by hypothesis, $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ so that $\bar{n}_{k_i}(r, a_i, f_1) = \bar{n}_{k_i}(r, a_i, f_2) = n_0^{(k_i)}(r, a_i)$ for $i = 1, 2, \ldots, m$.

The sequence $\langle k/(k+1) \rangle$ is increasing and so, since $k_1 \geq k_2 \geq \ldots \geq k_m$, (3) yields

$$\left\{ \sum_{i=1}^{m} \frac{k_i}{k_i + 1} - 2 \right\} \left\{ T(r, f_1) + T(r, f_2) \right\}
\leq \frac{2k_1}{k_1 + 1} \sum_{i=1}^{m} N_0^{(k_i)}(r, a_i) + S(r, f_1) + S(r, f_2)$$

(4)
Now, since \( f_1 \equiv f_2 \), it follows that, for \( a \in C \), each common zero of \( f_1 - a \) and \( f_2 - a \) is a pole of \( 1/(f_1 - f_2) \). Since \( a_1, \ldots, a_m \) are distinct, we therefore have

\[
\sum_{i=1}^{m} N_{0}^{(k_i)}(r, a_i) \leq N\left(r, \frac{1}{f_1 - f_2}\right) \leq T(r, f_1 - f_2) + O(1) \\
\leq T(r, f_1) + T(r, f_2) + O(1) .
\]

Hence, from (4), we obtain

\[
\left\{ \sum_{i=2}^{m} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} - 2 \right\} \{ T(r, f_1) + T(r, f_2) \} \\
\leq S(r, f_1) + S(r, f_2),
\]

which implies (2), as, otherwise, (5) would yield

\[
T(r, f_1) + T(r, f_2) = o(T(r, f_1) + T(r, f_2))
\]
as \( r \to \infty \) outside a set of finite measure, which is impossible.

Suppose, now, that some \( a_i \) is \( \infty \). Then, let \( a \) be a (finite) complex number different from \( a_1, \ldots, a_m \). Then \( 1/(a_1 - a), \ldots, 1/(a_m - a) \) are all distinct and finite. If \( g_j = 1/(f_j - a) \) for \( j = 1, 2 \), then \( g_1, g_2 \) are distinct meromorphic functions and

\[
E\left( \frac{1}{a_i - a}, k_i, g_1 \right) = E\left( \frac{1}{a_i - a}, k_i, g_2 \right)
\]

for \( i = 1, 2, \ldots, m \). Hence, by what we have proved above, (2) holds.

This completes the proof of Theorem 1.

**Consequences of Theorem 1.** Let \( f_1, f_2 \) be meromorphic functions.

(i) Suppose that there exist seven distinct elements \( a_1, \ldots, a_7 \) in \( \mathbb{C} \) such that \( E(a_i, k_i, f_1) = E(a_i, k_i, f_2) \) for \( i = 1, \ldots, 7 \), where each \( k_i \) is either a positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_7 \) and \( k_2 \geq 2 \) if \( k_1 = 0 \). Then \( k_1/(k_1 + 1) \leq 1 \) with equality holding only when \( k_1 = \infty \) and \( k_i/(k_1 + 1) \geq \frac{1}{2} \) for \( i = 2, \ldots, 7 \) with \( k_2/(k_2 + 1) \geq \frac{3}{2} \) if \( k_1 = \infty \).

Hence

\[
\sum_{i=2}^{7} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} \geq 2 .
\]

Hence by Theorem 1, \( f_1 \equiv f_2 \).

In particular, with \( k_1 = \ldots = k_7 = 1 \), it follows that if the set of simple zeros of \( f_1 - a \) coincides with the set of simple zeros of \( f_2 - a \) for seven distinct values of \( a \) in \( \mathbb{C} \) then \( f_1 \equiv f_2 \).
(ii) If there exist six distinct elements \(a_1, \ldots, a_6\) in \(\mathbb{C}\) such that \(E(a_i, k_i, f_1) = E(a_i, k_i, f_2)\) for \(i = 1, \ldots, 6\) where each \(k_i\) is a positive integer or \(\infty\) with \(k_1 \geq k_2 \geq \cdots \geq k_6, \ k_3 \geq 2\) and
\[
\frac{k_1}{k_1 + 1} < \frac{k_2}{k_2 + 1} + \frac{1}{6}
\]
(which holds, in particular, if \(k_1 = k_2\)) then
\[
\sum_{i=2}^{6} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} > 2.
\]
Hence, by Theorem 1, \(f_1 \equiv f_2\).

(iii) If there exist five distinct elements \(a_1, \ldots, a_5\) in \(\mathbb{C}\) such that \(E(a_i, k_i, f_1) = E(a_i, k_i, f_2)\) for \(i = 1, \ldots, 5\) where each \(k_i\) is a positive integer or \(\infty\) with \(k_1 \geq k_2 \geq \cdots \geq k_5 \geq 2, \ k_3 \geq 3\) and
\[
\frac{k_1}{k_1 + 1} < \frac{k_2}{k_2 + 1} + \frac{1}{12}
\]
(which holds if \(k_1 = k_2\), then
\[
\sum_{i=2}^{5} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} > 2
\]
and so \(f_1 \equiv f_2\) by Theorem 1.
This includes Theorem A of Nevanlinna mentioned earlier.

(iv) If there exist five distinct elements \(a_1, \ldots, a_5\) in \(\mathbb{C}\) such that \(E(a_i, k_i, f_1) = E(a_i, k_i, f_2)\) for \(i = 1, \ldots, 5\) where each \(k_i\) is a positive integer or \(\infty\) with \(k_1 \geq k_2 \geq \cdots \geq k_5, \ k_4 \geq 4\) and
\[
\frac{k_1}{k_1 + 1} < \frac{k_2}{k_2 + 1} + \frac{1}{10},
\]
then, again
\[
\sum_{i=2}^{5} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} > 2.
\]
and so \(f_1 \equiv f_2\).
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