INVOLUTION-INVARIANT GEODESICS

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To Werner Fenchel on his 70th birthday.

Introduction.

Given an isometry $A : M \to M$ on a compact, connected Riemannian
manifold $M$, the two most natural questions concerning $A$-invariant
geodesics on $M$ seems to be: (1) Are there any? and if so (2) How many?
In [6] and [7], we have developed a theory which gives rather satisfactory
answers to question (1). If e.g. $A$ is an involution and $M$ is simply
connected, then $A$ has a (non-trivial) invariant geodesic; obvious examples
(rotation or reflexion on $S^2$) shows however that $A$ does not in general
have more than one invariant geodesic. The strategy in deriving the
existence results in [6] and [7] was to assume, that $A$ has no invariant
gedesics. In the same spirit we shall here assume, that $A$ has only finitely
many invariant geodesics. We obtain thereby a topological condition
which ensures the existence of infinitely many closed $A$-invariant geo-
desics, when $A^2 = 1_M$. Besides [6] and [7], the theory developed in Gro-

1. Preliminaries.

Throughout the paper $M$ shall denote a connected, compact Riemann-
nian manifold and $A : M \to M$ an isometry on $M$. Recall that a geodesic
$\gamma : R \to M$ is said to be $A$-invariant if and only if there is a $\theta \geq 0$ such that
$A(\gamma(t)) = \gamma(t + \theta)$ for all $t \in R$. Note that if $p_0 \in M$ belongs to the fixed
point set, $\text{Fix}(A)$ of $A$, then $\gamma(t) = p_0$ for all $t \in R$ is an $A$-invariant geo-
desic; such a geodesic is called a trivial $A$-invariant geodesic. Whenever
we say $A$-invariant geodesic we think of a non-trivial $A$-invariant geo-
desic.

Let $L^2(I, M)$ denote the Hilbert manifold consisting of absolutely
continuous paths, $\sigma : I = [0, 1] \to M$ with square integrable derivative, $\dot{\sigma}$.
The tangent space to $L^2(I, M)$ at $\sigma$ consists of absolutely continuous
vectorfields, $X$ along $\sigma$ with square integrable covariant derivative $X'$.
The Riemannian structure on $M$ induces in a natural way a complete

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Riemannian structure on $L_1^2(I, M)$. If $X$ and $Y$ are tangent vectors at $\sigma$ we define
\[
\langle X, Y \rangle_1 = \langle X, Y \rangle_0 + \langle X', Y' \rangle_0,
\]
where $\langle X, Y \rangle_0 = \int_0^1 \langle X(t), Y(t) \rangle_{\sigma(t)} \, dt$ is the $L^2$-inner product (see Flaschel and Klingenberg [3]). In order to study $A$-invariant geodesics we introduced in [6] the closed submanifold
\[
A_A(M) = \{ \sigma \in L_1^2(I, M) \mid \sigma(1) = A(\sigma(0)) \}.
\]
The reason for this is that the critical points for the energy integral
\[
E: A_A(M) \to \mathbb{R}; \quad \sigma \to \frac{1}{2} \| \dot{\sigma} \|_0^2
\]
are exactly the geodesics, $c: I \to M$ with $\dot{c}(1) = A_* \dot{c}(0)$. Here $A_*: TM \to TM$ denotes the induced map of $A$. Thus, the existence of $A$-invariant geodesics on $M$ is equivalent to the existence of positive critical $E$-values. Since $E: A_A(M) \to \mathbb{R}$ satisfies condition (C) of Palais and Smale (see [6]), there is a rich critical point theory available for us.

Note that $A_A(M)$ can be identified with the set of locally $L_1^2$-maps, $\tilde{\sigma}: \mathbb{R} \to M$ satisfying $\tilde{\sigma}(t+1) = A(\tilde{\sigma}(t))$ for all $t \in \mathbb{R}$. The action of $\mathbb{R}$ on the parameter induces in this way a continuous action on $A_A(M)$ by isometries (see [7]). $E$ is clearly invariant and the orbit of a critical point is an immersed critical submanifold.

Assume from now on that $A$ has finite order i.e. $A^k = 1_M$ for some $k \in \mathbb{N}$. The $\mathbb{R}$-action reduces then to an $S^1$-action and each critical $S^1$-orbit is an imbedded critical submanifold of $A_A(M)$. By the index, $\lambda(c)$ and nullity, $\nu(c)$ of a critical point, $c$ we mean the index and nullity of the corresponding $S^1$-orbit as a critical submanifold. The Hessian of $E$ at a critical point $c$ is given by
\[
H(E)(X, Y) = \langle X', Y' \rangle_0 - \langle R(X, \dot{c}) \dot{c}, Y \rangle_0,
\]
where $R$ is the Riemannian curvature tensor of $M$. It follows from this and the fact that the inclusion $L_1^2 \subset L^2$ is compact, that the corresponding selfadjoint operator, $S$ determined by $H(E)(X, Y) = \langle SX, Y \rangle_1$ admits a decomposition $S = \text{Id} + K$, where $K$ is a compact operator (for an explicit expression for $K$ compare Eliasson [2]).

We shall now collect what we need from equivariant degenerate Morse theory. In finite dimensions, Morse defined for any isolated critical point a local homological invariant. This was modified and generalized to infinite dimensions by Gromoll and Meyer [4] under the above assumption on $S$ (and condition (C)).

Consider an isolated critical orbit $S^1 \cdot c$. Choose a small tubular neighbourhood of $S^1 \cdot c$, and let $E_\varepsilon$ denote the energy restricted to the fiber over
c. From the splitting lemma of Gromoll and Meyer [4] (compare also [5]) it follows that $E_c$ satisfies condition (C) and has only $c$ as critical point. Thus we have a well-defined local homological invariant of $E_c$ at $c$,

$$\mathcal{H}(E, c) = H_\#(W_c, W_c^-),$$

where $(W_c, W_c^-)$ is a pair of so called admissible regions (see [4, §2]). Although it is not necessary here, we take homology with coefficient in a field of characteristic zero. From this we obtain a well defined local homological invariant associated to the orbit $S^1 \cdot c$ as follows

$$\mathcal{H}(E, S^1 \cdot c) = H_\#(W, W^-),$$

where $W = S^1 \cdot W_c$ and $W^- = S^1 \cdot W_c^-$. The crucial property of this invariant is contained in the following lemma, which is proved exactly as Lemma 4 in Gromoll and Meyer [5].

**Lemma 1.2.** If $b$ is the only critical value of $E$ in $[b - \varepsilon, b + \varepsilon]$ for some $\varepsilon > 0$ and $S^1 \cdot c_1, \ldots, S^1 \cdot c_n$ the only critical orbits in $E^{-1}(b)$, then

$$H_\#(A_A(M)^{b+\varepsilon}, A_A(M)^{b-\varepsilon}) = \oplus_{i=1}^n \mathcal{H}(E, S^1 \cdot c_i),$$

where $A_A(M)^a$ as usual denotes $E^{-1}((-\infty, a])$.

In order to use this lemma we must control $\mathcal{H}(E, S^1 \cdot c)$. The isotropy group at $c$,

$$S_c^1 = \{ z \in S^1 \mid z \cdot c = c \}$$

operates by covering transformations on $S^1 \times (W_c, W_c^-)$ with quotient $(W, W^-)$. Hence

$$\mathcal{H}_k(E, S^1 \cdot c) = \mathcal{H}_{k-1}(E, c) \oplus \mathcal{H}_k(E, c).$$

In Gromoll and Meyer [4, §3] there was also introduced a characteristic invariant, $\mathcal{H}^0$, which is determined by the degenerate part of the function. This characteristic invariant together with the index, $\lambda$ of $c$ determines $\mathcal{H}$ completely by the shifting theorem:

$$\mathcal{H}_{k+\lambda}(E, c) = \mathcal{H}_k^0(E, c).$$

We shall sometimes omit $E$ in $\mathcal{H}(E, c)$ as well as in $\mathcal{H}^0(E, c)$.

All in all we can now see from (1.2–1.4) how information about the index and characteristic invariant of the critical points in $A_A(M)$ can give us information about the homology of $A_A(M)$. The problem here is of course that for each $A$-invariant geodesic, we have a whole tower of
different critical orbits in $A_\mathbb{A}(M)$. In the next sections we shall see how this problem can be treated when $A$ is an involution.

2. Translated geodesics.

Although some of the constructions in this section can be carried out in general, we assume from now on that $A^2 = 1_M$.

The $S^1$-action from Section 1 takes the form:

$$
\mu : S^1 \times A_\mathbb{A}(M) \to A_\mathbb{A}(M);
$$

$$
\mu(e^{\pi i \theta}, \sigma)(t) = \begin{cases} 
\sigma(t + \theta), & t \in [0, 1 - \theta] \\
A(\sigma(t + \theta - 1)), & t \in [1 - \theta, 1]
\end{cases}
$$

The critical points $c \in A_\mathbb{A}(M)$ are classified according to their isotropy groups $S^1_c$ in the following sense. $S^1_c = S^1$ if and only if $c$ is a trivial $A$-invariant geodesic. Otherwise $S^1_c = \mathbb{Z}_p$ for some $p \in \mathbb{N}$. Furthermore $p = 2l$ means that $c$ is an $l$-fold covering of a prime closed geodesic fixed by $A$. Similarly $p = 2l + 1$ means that $c$ contains an $l$-fold covering of a prime closed geodesic, on which $A$ operates as "the antipodal map" (and $l$ is maximal with this property). There arises in this way to each prime closed $A$-invariant geodesic a tower of critical orbits in $A_\mathbb{A}(M)$ corresponding to the coverings of the prime closed geodesic. If an $A$-invariant geodesic is not fixed by $A$ we say that it is translated. A translated geodesic is now clearly characterized by the fact that all isotropy groups in the corresponding tower of critical orbits have odd order. Fixed geodesics are characterized analogously.

We shall now in some sense reduce the study of translated geodesics to that of closed geodesics.

Let $\mathbb{Z}_2$ act on $M$ by $A$ and on $S^1$ by the antipodal map. The product action is a free isometric $\mathbb{Z}_2$-action on $M \times S^1$. Note that the quotient manifold,

$$
M \times A S^1 \equiv M \times S^1 / \mathbb{Z}_2
$$

is the usual mapping torus of $A$, $M \times I/(p, 0) \sim (A(p), 1)$.

Complete information about $M \times A S^1$ is contained in the diagram

$$
\begin{array}{ccc}
M & \to & M \\
\downarrow & & \downarrow \\
M \times S^1 & \to & M \times A S^1 \\
pr_2 & \downarrow & \pi \\
S^1 & \to & S^1,
\end{array}
$$
where the horizontal maps are $Z_2$-coverings. In particular we see that the fiber bundle $\pi: M \times_A S^1 \rightarrow S^1$ is locally isometric to a product. A geodesic in $M \times_A S^1$ is therefore locally a product of a geodesic in $M$ with a geodesic in $S^1$.

The space of closed curves on $M \times_A S^1$, denoted $\Lambda(M \times_A S^1)$, contains $\Lambda_A(M)$ as a submanifold. More precisely

$$F: \Lambda_A(M) \rightarrow \Lambda(M \times_A S^1)$$

defined by $F(\sigma) = \bar{\sigma}$, where $\bar{\sigma}(t) = (\sigma(t), t)$ for $t \in I/0 \sim 1$, is an isometric imbedding onto the space of $L^2$-sections of $\pi: M \times_A S^1 \rightarrow S^1$. In order to describe the towers of critical orbits in $\Lambda_A(M)$ and $\Lambda(M \times_A S^1)$ we define for any positive integer $m$ the iteration maps

$$m: \Lambda_A(M) \rightarrow \Lambda_A m(M) ; \quad \sigma \rightarrow \sigma^m$$

and

$$m: \Lambda(M \times_A S^1) \rightarrow \Lambda(M \times_A S^1) ; \quad \sigma \rightarrow \sigma^m$$

where $\sigma^m(t) = \bar{\sigma}(m \cdot t)$ for all $t \in I$. Note that $E(\sigma^m) = m^2 E(\sigma)$ and that $m$ is an imbedding. For each translated geodesic, $\gamma$ there exists a critical point $c \in \Lambda_A(M)$ with $S^1_c = \mathbb{Z}_2 = \{0\}$, such that the collection $\{c^m | m \text{ odd}\}$ represents the tower of critical orbits associated with $\gamma$.

**Lemma 2.1.** For any odd $m \in \mathbb{N}$ we have with the above notation $\lambda(c^m) = \lambda(\bar{c}^m)$ and $\nu(c^m) + 1 = \nu(\bar{c}^m)$.

**Proof.** Since $m$ is odd, $c^m$ (respectively $\bar{c}^m$) is clearly a critical point in $\Lambda_A(M)$ (respectively $\Lambda(M \times_A S^1)$). The tangent space at an arbitrary point $s \in \Lambda(M \times_A S^1)$ admits a splitting

$$T_s \Lambda(M \times_A S^1) = T^v_s \oplus T^h_s$$

in vertical and horizontal vectors. In this splitting $X = X^v + X^h$, $X^v$ is tangent to the fiber $M$ along $s$ and $X^h$ is pointwise orthogonal to $X^v$. If $s$ is a critical point in $\Lambda(M \times_A S^1)$ we get (compare (1.1))

$$H(E)_s(X, Y) = \langle X', Y' \rangle_0 - \langle R(X, \delta) \delta, Y \rangle_0$$

$$= \langle X^v', Y^v \rangle_0 - \langle R^v(X^v, \delta^v) \delta^v, Y^v \rangle_0$$

$$+ \langle X^h', Y^h \rangle_0 ,$$

where $R$ denotes the curvature tensor on $M \times_A S^1$ and $R^v$ is the curvature tensor on $M$ (we have used that $M \times_A S^1$ is locally isometric to $M \times S^1$). Vertical and horizontal vectors are therefore orthogonal with respect to the Hessian i.e. $\lambda(s) = \lambda v(s) + \lambda h(s) = \lambda v(s)$ and $\nu(s) = \nu v(s) + \nu h(s) = \nu v(s) + 1$.  


To finish the proof, we only have to note that the tangent space at $c^m$ is canonically identified with the vertical vectors at $\overline{c^m}$.

Lemma 2.1 reduces the study of the index and nullity of the translated geodesics to the corresponding study of closed geodesics. Thus Lemma 1 and Lemma 2 in [5] give us immediately

**Lemma 2.2.** Either $\lambda(c^m) = 0$ for all (odd) $m$ or there exist numbers $\varepsilon > 0$ and $a > 0$ such that

$$\lambda(c^{m+s}) - \lambda(c^m) \geq s \cdot \varepsilon - a$$

for all (odd) $m$ and (even) $s$.

**Lemma 2.3.** There are positive (odd) integers $k_1, \ldots, k_s$ and (odd) sequences $m_j^i \in \mathbb{N}, i > 0, j = 1, \ldots, s$ with $m_j^1 = 1$ such that

$$\nu(c^{m_j^i k_j}) = \nu(c^{k_j})$$

and such that any (odd) $m$ with $\nu(c^m) \neq 0$ can be written uniquely as $m = m_j^i k_j$.

This lemma enables us to get the desired information about the characteristic invariants. Since $m_j^i$ is odd we have the iteration map $m_j^i : \Lambda_\Delta(M) \to \Lambda_\Delta(M)$. We can therefore proceed in complete analogy with Lemma 5 and Theorem 3 in Gromoll and Meyer [5], whereby we obtain

**Lemma 2.4.** With $k_j$ and $m_j^i$ as in Lemma 2.3 we have

$$\mathcal{H}^0(c^{m_j^i k_j}) = \mathcal{H}^0(c^{k_j}).$$

We see in particular, that there are at most finitely many different characteristic invariants associated to the tower of critical orbits corresponding to a translated geodesic.

3. Fixed geodesics.

In this section $c$ will denote a prime closed geodesic which is fixed by $A$, that is, $c$ is a critical point in $\Lambda_\Delta(M)$ with isotropy group $S^1_c = \mathbb{Z}_2$. The corresponding tower of critical orbits is represented by all the iterates of $c$. We must therefore consider $c^m$ for all positive integers $m$. Thus proceeding as in Section 2 we would have to allow also for even values of $m_j^i$ and $k_j$. This would then give information about the index and nullity of $c^{m_j^i k_j}$ in $\Lambda(M) = \Lambda_\Delta(M)$ and not as desired in $\Lambda_\Delta(M)$. 
To overcome these difficulties recall that $\text{Fix}(A)$ is a disjoint union of closed totally geodesic submanifolds of $M$ (see e.g. [6]). Hence $\Lambda(\text{Fix}(A))$ is a disjoint union of closed totally geodesic submanifolds of $\Lambda_A(M)$. Denote by $\lambda^T(c^m)$ (respectively $\nu^T(c^m)$) the index (respectively nullity) of $c^m$ in $\Lambda(\text{Fix}(A))$. Note that the normal space $T_\sigma \Lambda(\text{Fix}(A))^\perp$ of $\Lambda(\text{Fix}(A))$ at $\sigma \in \Lambda(\text{Fix}(A))$, consists of vectors $X \in T_\sigma \Lambda_A(M)$ with $X$ pointwise orthogonal to $\text{Fix}(A)$. Using that $\text{Fix}(A)$ is totally geodesic together with the symmetry properties of the curvature tensor we get from (1.1).

**Lemma 3.1.** The above splitting $T_{c^m} \Lambda_A(M) = T_{c^m}^T \oplus T_{c^m}^\perp$ is orthogonal with respect to the Hessian at $c^m$. In particular

$$\lambda(c^m) = \lambda^T(c^m) + \lambda^\perp(c^m)$$

and

$$\nu(c^m) = \nu^T(c^m) + \nu^\perp(c^m),$$

where $\lambda^\perp(c^m)$ (respectively $\nu^\perp(c^m)$) denotes the index (respectively nullity) of the Hessian restricted to the normal space, $T_{c^m}^\perp$ at $c^m$.

A study of $\lambda^T(c^m)$ (respectively $\nu^T(c^m)$) has as we know from Section 2 already been carried out by Bott [1], and Gromoll and Meyer [5]. It remains to study $\lambda^\perp(c^m)$ (respectively $\nu^\perp(c^m)$). We observe that the results in Bott [1] can be used here as well. Consider the complexification $L$ of the index operator

$$\mathcal{L}(X) = X'' + R(X, \dot{c})\dot{c}$$

and let $L^\perp$ (respectively $L^T$) denote the restriction of $L$ to the normal (respectively tangent) space. It now follows that $\lambda^\perp(c^m)$ and $\nu^\perp(c^m)$ are completely determined by functions $A^\perp : S^1 \to \mathbb{Z}^+ \cup \{0\}$ and $N^\perp : S^1 \to \mathbb{Z}^+ \cup \{0\}$ by the formulas

$$\lambda^\perp(c^m) = \sum_{z^m = -1} A^\perp(z)$$
$$\nu^\perp(c^m) = \sum_{z^m = -1} N^\perp(z).$$

(3.2)

(Note that $X \in T_{c^m}^\perp$ satisfies $X(1) = A_* X(0) = -X(0)$ and compare with Corollary p. 178 in [1]). In the corresponding formulas for $\lambda^T$ and $\nu^T$ we sum functions $A^T$ and $N^T$ over $m$-roots of 1. The following quite elementary facts are contained in Proposition 1.3 of Bott [1].

$$A^\perp(z) = A^\perp(\bar{z}) \quad \text{and} \quad N^\perp(z) = N^\perp(\bar{z}).$$

$N^\perp(z) = 0$ except for at most $2 \cdot \text{codim Fix}(A)$ points, the Poincaré points of $L^\perp$. 

$$N^\perp(z) = 0$$

(3.3) \( A^\perp \) is locally constant except at Poincaré points, where the jump is at most the absolute value of \( N \).

\[
\lim_{z \to z_0 \pm} A^\perp(z) \geq A^\perp(z_0).
\]

As in Lemma 1 of [5] we use these facts together with the corresponding ones for \( A^T \) (derived from the properties of \( L^T \)) to obtain.

**Lemma 3.4.** Either \( \lambda(c^m) = 0 \) for all \( m \) or there are numbers \( \epsilon > 0 \) and \( a > 0 \) such that

\[
\lambda(c^{m+s}) - \lambda(c^m) \geq s \cdot \epsilon - a
\]

for all \( m \) and \( s \).

As far as the index is concerned we have just seen how we could study \( \lambda^T \) and \( \lambda^\perp \) separately. However, in order to study the characteristic invariant, we must study \( \nu^T \) and \( \nu^\perp \) simultaneously. Denote by \( P^\perp \) (respectively \( P^T \)) the Poincaré points for \( L^\perp \) (respectively \( L^T \)). These points can be described by means of the geodesic flow in the unit tangent bundle, \( T_1M \) of \( M \). The closed orbit in \( T_1M \) corresponding to \( c \) is simply the velocity vector field of \( c \) parametrized by arch length. Let \( P_c \) be the differential of the Poincaré map associated to this orbit. In the splitting of \( TTM \) in horizontal and vertical subbundles (induced by the Riemannian connection on \( M \)), \( P_c \) takes the following form:

\[
P_c(u,v) = (Y(1), Y'(1)),
\]

where \( Y \) is the unique Jacobi field orthogonal to \( c \) with \( (Y(0), Y'(0)) = (u,v) \). Note also that \( A_{**} = (A_*, A_*) \) in the splitting

\[
TTM = T^H T M \oplus T^V T M.
\]

It is now clear that \( P_c \) commutes with \( A_{**} \). Hence \( P_c \) preserves the \((+)1\)-eigenspace of \( A_{**} \). The set of eigenvalues with absolute value 1 of \( P_c \) (complexified) restricted to the \((+)1\)-eigenspace of \( (A_*, A_*) \) is now exactly \( P^\perp \) (respectively \( P^T \)). The crucial information about the nullity of \( c^m \) is contained in

**Lemma 3.5.** There are positive integers \( k_1, \ldots, k_s \) and sequences \( m_j^i \in \mathbb{N}, \ i > 0, j = 1, \ldots, s \) with \( m_j^1 = 1 \) such that

\[
\nu(c^{m^i_1 k}) = \nu(c^{k_j}), \quad m_j^i \text{ odd}
\]

\[
\nu(c^{m^i_1 k}) = \nu^T(c^{m^i_1 k}) = \nu^T(c^{k_j}), \quad m_j^i \text{ even}
\]

and such that any \( m \) with \( \nu(c^m) \neq 0 \) can be written uniquely as \( m = m_j^i k_j \).
Proof. It follows from (3.2) and (3.3) that $\nu^\perp(c^m) \neq 0$ if and only if there is a $z = e^{2\pi i q} \in P^T$ with $z^q = -1$. This implies that $q$ is rational with even denominator. Similarly $\nu^\perp(c^m) \neq 0$ if and only if there is a $z' = e^{2\pi i q'} \in P^L$ with $z'^q = 1$ (which implies that $q'$ is rational). Let now $Q = Q^T \cup Q^L$, where

$$Q^T = \{ q \in \mathbb{N} \mid \exists z' = \exp(2\pi i p/q) \in P^T, (p, q) = 1 \}$$

and

$$Q^L = \{ q \in \mathbb{N} \mid \exists z = \exp(2\pi i p/2q) \in P^L, (p, 2q) = 1 \}.$$

For every $D \subset Q$ let $k(D)$ denote the least common multiple of elements in $D$. Choose distinct numbers $k_1, \ldots, k_n$ such that for each $D \subset Q$ there is a $j$ with $k_j = k(D)$. For each fixed $k_j$ select from the sequence $m k_j$, $m > 0$ the greatest subsequence $m^j k_j$, $i > 0$, with the property that whenever $q \in Q$ and $q|m^j k_j$ then $q|k_j$. If

$$\nu(c^m) = \nu^T(c^m) + \nu^L(c^m) \neq 0$$

there is a $q \in Q$ which divides $m$; Let $D = \{ q \in Q \mid q|m \}$ and $k(D) = k_j$, then $m = m^j k_j$ is unique by construction. It is now easy to see that

$$\{ z \in P^T \mid z^{m^j k_j} = 1 \} = \{ z \in P^T \mid z^{k_j} = 1 \}$$

and therefore by (3.2) and (3.3) (or rather the corresponding formulas for $N^T$)

$$\nu^T(c^{m^j k_j}) = \nu^T(c^{k_j})$$

for all $m^j k_j$ and $k_j$. The same statement for $\nu^L$ is false! However, if $\nu^L(c^{m^j k_j}) \neq 0$, we see from (3.2) and (3.3) that $m^j$ must be odd and furthermore

$$\nu^L(c^{m^j k_j}) = \nu^L(c^{k_j}).$$

In consequence of this lemma we obtain the needed information about the characteristic invariants.

Lemma 3.6. With $k_j$ and $m^j$ as in Lemma 3.5 we have

for $m^j$ odd: $\mathcal{H}^0(c^{m^j k_j}) = \mathcal{H}^0(c^{k_j})$

and

for $m^j$ even: $\mathcal{H}^0(c^{m^j k_j}) = \mathcal{H}^0(c^{k_j})^T$,

where $\mathcal{H}^0(c^{k_j})^T$ denotes the characteristic invariant of $c^{k_j}$ in the space $\Lambda(\text{Fix}(A)).$

Proof. for $m^j$ odd we use Lemma 3.5 and proceed exactly as in Section 2 ($m^j: \Lambda_A(M) \to \Lambda_A(M)$). If $m^j$ is even $\nu^L(c^{m^j k_j}) = 0$ by Lemma
3.5. In other words the null-space of the Hessian at $c^{m,i,j}$ is contained in the tangent space to $\Lambda(\text{Fix}(A))$. Furthermore $\text{grad}E$ is tangent to $\Lambda(\text{Fix}(A))$. To see this let $\tilde{\sigma} \in \Lambda(\text{Fix}(A)) \subset \Lambda_\sigma(M)$ and assume without loss of generality that $\tilde{\sigma}$ is $C^\infty$. For $X \in T_{\tilde{\sigma}}$ we have (see [6] or [3])

$$dE(X) = \langle X', \dot{\tilde{\sigma}} \rangle_0 = -\langle X, \dot{\tilde{\sigma}}' \rangle_0$$

which is equal to zero since $\text{Fix}(A)$ is totally geodesic. Thus Lemma 7 in Gromoll and Meyer [4] tells us that

$$\mathcal{H}^0(E, c^{m,i,j}) = \mathcal{H}^0(\mathcal{F}|_{\Lambda(\text{Fix}(A))}, c^{m,i,j})$$

and therefore by construction (see Section 1)

$$\mathcal{H}^0(c^{m,i,j}) = \mathcal{H}^0(c^{m,i,j})^T.$$

To finish the proof, we note that $\nu^T(c^{m,i,j}) = \nu^T(c^{k,j})$ by Lemma 3.5 and hence

$$\mathcal{H}^0(c^{m,i,j})^T = \mathcal{H}^0(c^{k,j})^T$$

by Theorem 3 in [5] applied to $c^{k,j} \in \Lambda(\text{Fix}(A))$.

The conclusion of this section is the same as that of Section 2: There are at most finitely many different characteristic invariants associated to the tower of critical orbits corresponding to a fixed geodesic.

4. Existence of infinitely many invariant geodesics.

The finiteness of the characteristic invariants together with the growth estimate of the indices is sufficient information to prove our main theorem:

**Theorem 4.1.** Let $M$ be a compact, 1-connected Riemannian manifold and $A: M \to M$ an involutive isometry on $M$. Then $A$ has infinitely many closed invariant geodesics if the sequence of Betti numbers

$$\beta_k(\Lambda_\sigma(M)) = \dim H_k(\Lambda_\sigma(M)),$$

$k \geq 0$ is unbounded.

**Proof.** Suppose $A$ has at most finitely many different (prime closed) invariant geodesics. Each critical orbit in $\Lambda_\sigma(M)$ is then isolated and we can therefore apply our results in the previous sections.

Fix a prime closed $A$-invariant geodesic represented by $c \in \Lambda_\sigma(M)$ ($\mathcal{S}_c = \mathbb{Z}_1$ or $\mathbb{Z}_2$) and consider the corresponding tower of critical orbits in $\Lambda_\sigma(M)$. Set

$$B_k(c^m) = \dim \mathcal{H}_k(c^m) \quad \text{and} \quad B^0_k(c^m) = \dim \mathcal{H}^0_k(c^m).$$
From Lemma 2.4 or Lemma 3.6 we can find a $B > 0$ such that $B_k^o(c^m) \leq B$ for all $k$ and $m$. Furthermore by definition $B_k^o(c^m) = 0$ for $k > 2(\dim M - 1)$ and all $m$. Thus from this, (1.3) and (1.4) we see that $B_k(c^m)$ is uniformly bounded by $2B$. Moreover, (1.3) and (1.4) in connection with Lemma 2.2 or Lemma 3.4 shows, that the number of orbits, $S^1 \cdot c^m$ with $B_k(c^m) \neq 0$ is bounded by some constant $C > 0$ independent of $k \geq 2 \dim M$.

Since there are only finitely many towers $\{S^1 \cdot c^m_i\}, i = 1, \ldots, r$ of critical orbits, we may assume without loss of generality that $B_k(c^m_i) \leq 2B$ for all $i$ and $m$ and that the number of orbits, $S^1 \cdot c^m_i$ with $B_k(c^m_i) \neq 0$ is bounded by $r \cdot C$ for all $k \geq 2 \dim M$. Hence with $K = 2B \cdot r \cdot C$ we get from Lemma 1.2 together with an exact sequence argument,

$$\beta_k(A_\Delta(M)^b, A_\Delta(M)^c) \leq K, \quad k \geq 2 \dim M,$$

where $a, b \in \mathbb{R}^+$ are arbitrary regular $E$-values. According to Corollary 3.3 in [6] we can choose $a = e$ such that $\text{Fix}(A)$ is a strong deformation retract of $A_\Delta(M)^c$. With this choice,

$$\beta_k(A_\Delta(M)^b, A_\Delta(M)^c) = \beta_k(A_\Delta(M)^b, \text{Fix}(A))$$

and therefore

$$\beta_k(A_\Delta(M)^b) = \beta_k(A_\Delta(M)^b, \text{Fix}(A)) \leq K$$

for all $k \geq 2 \dim M$ and all regular values $b \in \mathbb{R}^+$. Fix now $k \geq 2 \dim M$ and choose $b$ so large that

$$\mathcal{H}_k(S^1 \cdot c^m_i) = 0 \quad \text{for all } c^m_i \in A_\Delta(M)/\Delta(A_\Delta(M)^b).$$

This is possible by (1.3), (1.4) and Lemma 2.2 or Lemma 3.4. For such a $b$ obviously $\beta_k(A_\Delta(M)) = \beta_k(A_\Delta(M)^b)$ and hence

$$\beta_k(A_\Delta(M)) \leq K, \quad \text{when } k \geq 2 \dim M.$$

We conclude that $\beta_k(A_\Delta(M)), k \geq 0$ is bounded if $A$ has only finitely many closed invariant geodesics.

**Remark.** As far as the topology of $A_\Delta(M)$ is concerned we may as well consider

$$C_\Delta^0(M) = \{f: I \to M \mid f \text{ continuous and } f(1) = A(f(0))\},$$

since $A_\Delta(M) \subset C_\Delta^0(M)$ is a homotopy equivalence by theorem 1.3 of [6]. It is very likely, that the recent De Rham homotopy theory of Sullivan can be used to decide when $A_\Delta(M)$ has unbounded Betti numbers. However so far the question has not been settled completely even in the case $A = 1_M$. 
REFERENCES


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