ON THE HOMOLOGY OF INTERSECTIONS OF
COMPLEX PROJECTIVE MANIFOLDS

MOGENS ESROM LARSEN

1. Statement of results.

1.1. This note is concerned with the homology and cohomology of a complex submanifold in a complex projective space, which occurs as an intersection of two high-dimensional complex submanifolds.

Let $\mathbb{P}_n$ denote the complex projective space of dimension $n$, and let $A \subseteq \mathbb{P}_n$ and $B \subseteq \mathbb{P}_n$ be submanifolds of dimensions $a$ and $b$ respectively. Suppose, that $2a \geq n + 1$ and $2b \geq n$, then $a + b > n$ and from [2, proposition 4], $A \cap B$ is connected. Suppose further that $A \cap B$ is a submanifold of $\mathbb{P}_n$. Throughout this note let $s = \min\{2b - n, 2a - n - 1\}$. The results are stated in 1.2. and 1.3.

1.2. Theorem 1. Let $A, B$ and $A \cap B$ be submanifolds of $\mathbb{P}_n$ and $\dim A = a$, $\dim B = b$, and $s = \min\{2b - n, 2a - n - 1\}$. Then the inclusion $A \cap B \subseteq B$ induces isomorphisms

$$H^i(B; \mathbb{Z}) \cong H^i(A \cap B; \mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \text{ odd} , \\ \mathbb{Z} & \text{for } i \text{ even} , \end{cases}$$

$$H_4(A \cap B; \mathbb{Z}) \cong H_4(B; \mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \text{ odd} , \\ \mathbb{Z} & \text{for } i \text{ even} , \end{cases}$$

for $i \leq s$. Further the relative groups satisfy

$$H^i(B, A \cap B; \mathbb{Z}) = 0 \quad \text{for } i \leq s + 1 ,$$

$$H_4(B, A \cap B; \mathbb{Z}) = 0 \quad \text{for } i \leq s + 1 .$$

1.3. Theorem 2. Under the conditions of theorem 1 and further $\pi_1(A \cap B) = 0$, the relative groups

$$\pi_4(B, A \cap B) = 0 \quad \text{for } i \leq s + 1 ,$$

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and the inclusion $A \cap B \subseteq B$ induces isomorphisms

$$\pi_i(A \cap B) \cong \pi_i(B) \cong \begin{cases} 0 & \text{for } i \neq 2, i \leq s, \\ \mathbb{Z} & \text{for } i = 2. \end{cases}$$

1.4. These results generalize the classical theorem of Lefschetz, when $A$ is a hypersurface in $\mathbb{P}_n$, cf. [5, § 7].

2. The Hopf fibration.

2.1 $\mathbb{P}_n$ is the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$ and

$$S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}.$$ 

The Hopf fibration $h: S^{2n+1} \to \mathbb{P}_n$ is the restriction of the obvious map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n$.

If $X \subseteq \mathbb{P}_n$, we put $\hat{X} = h^{-1}(X) \subseteq S^{2n+1}$. The space $\hat{X}$ is the total space in a fiber bundle over $X$ with fiber $S^1$.

2.2. The following fact is well-known. For $X \subseteq \mathbb{P}_n$ there is a commutative diagram

$$\begin{array}{cccccccc}
\ldots & \to & \pi_m(S^{2n+1}) & \to & \pi_m(S^{2n+1}, \hat{X}) & \to & \pi_{m-1}(\hat{X}) & \to & \pi_{m-1}(S^{2n+1}) & \to & \ldots \\
\downarrow & & \downarrow & & \cong & & \downarrow & & \downarrow & & \pi_{m-1}(h) \\
\ldots & \to & \pi_m(\mathbb{P}_n) & \to & \pi_m(\mathbb{P}_n, X) & \to & \pi_{m-1}(X) & \to & \pi_{m-1}(\mathbb{P}_n) & \to & \ldots \\
\end{array}$$

The map $\pi_m(h)$ is an isomorphism for $m \neq 2$.

2.3. From 2.2. follows, that if $\pi_m(\hat{X}) = 0$, then $\pi_m(\mathbb{P}_n, X) = 0$ for $1 \leq m \leq s + 1$.

2.4. From 2.2. follows further, that if $\pi_1(X) = 0$, then $\pi_1(\hat{X})$ is abelian and hence isomorphic to $H_1(\hat{X}; \mathbb{Z})$. If further $H_m(\hat{X}; \mathbb{Z}) = 0$ for $1 \leq m \leq s$, then by the Hurewicz isomorphism theorem $\pi_m(\hat{X}) = 0$ for $1 \leq m \leq s$, and hence it follows from 2.3. that $\pi_m(\mathbb{P}_n, X) = 0$ for $1 \leq m \leq s + 1$.

2.5. Throughout this paper let $Z$ denote one of the groups $\mathbb{Z}$ or $\mathbb{Z}/p$ for $p$ prime. Let $D \subseteq E \subseteq \mathbb{P}_n$. From the general Gysin cohomology sequence

$$\ldots \to H^{m-2}(E, D; Z) \to H^m(E, D; Z) \to H^m(\hat{E}, \hat{D}; Z) \to H^{m-1}(E, D; Z) \to \ldots,$$

we deduce that if

$$H^{m-1}(E, D; Z) = H^m(E, D; Z) = 0$$

then $H^m(\hat{E}, \hat{D}; Z) = 0$. Also if $H^m(\hat{E}, \hat{D}; Z) = 0$ for $1 \leq m \leq s$ then

$$H^{m-2}(E, D; Z) \cong H^{m}(E, D; Z)$$
for $3 \leq m \leq s$. In the absolute case ($D = \emptyset$), we have that $H^m(\tilde{E}; Z) = 0$ for $1 \leq m \leq s$ implies that

$$H^{m-2}(E; Z) \to H^m(E; Z)$$

is an isomorphism for $2 \leq m \leq s$.

2.6. Let

$$\begin{array}{ccc}
\tilde{N} & \xrightarrow{\varphi} & S^{2n+1} \\
g & \downarrow & h \\
N & \xrightarrow{\varphi} & P_n
\end{array}$$

be a cartesian diagram. In particular $g : \tilde{N} \to N$ is a $S^1$-bundle. Let $\tilde{D} \subset \tilde{E}$ be a pair in $\tilde{N}$, and put $\varphi(\tilde{D}) = D$, $\varphi(\tilde{E}) = E$. Then the general Gysin cohomology sequences applied to $h : (\tilde{E}, \tilde{D}) \to (E, D)$ and $g : g^{-1}(\tilde{E}, \tilde{D})) \to (\tilde{E}, \tilde{D})$ gives a commutative diagram showing that if $\varphi : (\tilde{E}, \tilde{D}) \to (E, D)$ induces isomorphisms

$$H^m(E, D) \cong H^m(\tilde{E}, \tilde{D})$$

in all dimensions, then $\varphi : (g^{-1}(\tilde{E}), g^{-1}(\tilde{D})) \to (\tilde{E}, \tilde{D})$ induces isomorphisms

$$H^m(\tilde{E}, \tilde{D}) \cong H^m(g^{-1}(\tilde{E}), g^{-1}(\tilde{D}))$$

in all dimensions.

3. Construction of a ball $K$ in $SU(n+1)$.

3.1. Let $X$ be a complete Riemannian manifold and $C(y, \varrho)$ denote the closed ball of radius $\varrho$ around $y \in X$ with respect to the Riemannian metric, dist.

**Lemma 1.** Let $M \subset X$ be a compact $C^\infty$-submanifold. Then there exists a positive number $r = r(M)$, such that for $\varrho < r$ and $y \in X$ the intersection $C(y, \varrho) \cap M$ is either empty or homotopy equivalent to a point.

**Proof.** Let $T \subset X$ be a tubular neighbourhood of $M$ with tubular radius $r_0$. Then for all $y \in T$, there is only one $x \in M$, such that dist$(y, x) = \text{dist}(y, M)$, and only in this case the geodesic from $x$ to $y$ is orthogonal to $T_x M$. Put $r = r_0$ and let $\varrho < r$.

From [5, Lemma 10.3, p. 59] the function $f : X \to \mathbb{R}$ defined by

$$f(x) = \text{dist}(y, x)$$

is differentiable, and

$$C(y, \varrho) \cap M = \{ x \in M \mid f(x) \leq \varrho \}.$$
If \( x \in M \) is a critical point of \( f|_M \cap C(y, q) \), then the geodesic from \( x \) to \( y \) is orthogonal to \( T_x M \), so only one critical point can exist.

If \( C(y, q) \cap M \neq \emptyset \), then it follows from [5, Theorem 3.1, p. 12] that \( C(y, q) \cap M \) is homotopy equivalent to a point.

**Remark.** Obviously \( r(M) = r(u(M)) \), if \( u : X \to X \) is a transformation, preserving the Riemannian structure.

3.2. Let \( G \) denote the special unitary group \( SU(n+1) \). Then
\[
G \subseteq GL(n+1, \mathbb{C}) \subseteq \mathbb{C}^{(n+1)^2} = \mathbb{R}^m
\]
for \( m = 2(n+1)^2 \). Further the unitary group \( U(n+1) \) is embedded in \( \mathbb{C}^{(n+1)^2} \). Now any \( \sigma \in U(n+1) \) gives by matrix multiplication a map
\[
u \in \mathbb{C}^{(n+1)^2} \to \sigma u \in \mathbb{C}^{(n+1)^2},
\]
which preserves the euclidean distance in \( \mathbb{R}^m \).

3.3. Define \( K(q) \) as \( G \cap (C(1), q) \), where 1 is the unit matrix in \( GL(n+1, \mathbb{C}) \). Fix \( r > 0 \) so small, that the following two conditions are fulfilled.

1) For any \( y \in P_n \) let \( G_y \) be the subgroup of \( G \) fixing \( y \). By lemma 1 using \( G \) compact we can suppose for \( q < r \) that \( K(q) \cap \sigma \tau G_y \tau^{-1} \) is either empty or homotopy equivalent to a point for \( \sigma, \tau \in U(n+1) \).

2) Since \( \sigma K(q) \sigma^{-1} = K(q) \) for all \( \sigma \in U(n+1) \), and since \( U(n+1) \) operates doubly transitively on \( P_n \), there exists a function \( d(q) \), such that for all \( z \in P_n \),
\[
K(q)z = \{ x \in P_n \mid \text{dist}(z, x) \leq d(q) \}.
\]
Compare [1, Lemmata 1, 2, 3]. Choose \( r \) so small, that for all \( q \leq r \) the set \( K(q)A \) is a tubular neighbourhood of \( A \) in \( P_n \). Then obviously \( K(q)\sigma A \) are tubular neighbourhoods of \( \sigma A \) for all \( \sigma \in G \).

Put \( K = K(\frac{1}{2}r) \).

4. **Statement of lemmate 2 and 3.**

4.1. \( G \) operates transitively on \( S^{2n+1} \) and \( P_n \). The Hopf map \( h \) is \( G \)-equivariant. Let \( A \subseteq P_n \). We study the maps
\[
\hat{\varphi} : G \times \hat{A} \to S^{2n+1},
\]
\[
\varphi : G \times A \to P_n.
\]
defined by $\hat{\varphi}(\sigma, \hat{x}) = \sigma \hat{x}$ for $\sigma \in G$ and $\hat{x} \in \hat{A}$, and $\varphi(\sigma, x) = \sigma x$ for $\sigma \in G$ and $x \in A$. We have the commutative diagram

$$
\begin{array}{c}
G \times \hat{A} \rightarrow S^{2n+1} \\
\downarrow_{id \times (h|\hat{A})} \downarrow^h \\
G \times A \rightarrow P_n
\end{array}
$$

$G$ operates on $G \times \hat{A}$, respectively $G \times A$, by left translation on the first factor. With respect to this operation, $\hat{\varphi}$ and $\varphi$ are equivariant.

4.2. Let $K \subseteq G$ be chosen as in 3.3. For any $\sigma \in G$ we have a commutative diagram

$$
\begin{array}{c}
\{\sigma\} \times A \cap \varphi^{-1}(B) \\
\downarrow \\
K \sigma \times A \cap \varphi^{-1}(B)
\end{array} \rightarrow \sigma A \cap B
$$

with inclusions as vertical arrows and restrictions of $\varphi$ as horizontal arrows. The upper map is a homeomorphism.

4.3. $H^q(\varphi)$ is the map

$$
H^q(K \sigma A \cap B, \sigma A \cap B; Z) \rightarrow H^q(K \sigma \times A \cap \varphi^{-1}(B), \{\sigma\} \times A \cap \varphi^{-1}(B); Z)
$$

**Lemma 2.** $H^q(\varphi)$ is an isomorphism for all $q$.

**Remark.** Lemma 2 remains valid when $B$ is exchanged with $A \cap B$.

**Lemma 3.** $H^q(K \sigma A \cap B, \sigma A \cap B; Z) = 0$ for $0 \leq q \leq s$.

**Remark.** Lemma 3 remains valid when $B$ is exchanged with $A \cap B$.

4.4. **Lemma 2'.** $H^q(\hat{\varphi})$ is an isomorphism for all $q$.

**Proof.** Follows from 2.6.

**Lemma 3'.** $H^q(K \sigma \hat{A} \cap \hat{B}, \sigma \hat{A} \cap \hat{B}; Z) = 0$ for $0 \leq q \leq s$.

**Proof.** Follows from 2.5.

5. **Proof of lemma 2.**

5.1. The map

$$
\varphi: K \sigma \times A \cap \varphi^{-1}(B) \rightarrow K \sigma A \cap B
$$

is proper and surjective. In order to show, that $\varphi$ induces isomorphisms

$$
H^q(\varphi): H^q(K \sigma A \cap B; Z) \rightarrow H^q(K \sigma \times A \cap \varphi^{-1}(B); Z)
$$
for all $q$, it is by the Leray spectral sequences enough to show, that all fibers have the homotopy type of a point. If $y \in K\sigma A \cap B$, the fiber satisfies

$$\varphi^{-1}(y) = \{ (x, x) \in K\sigma \times B \mid \tau x = y \}$$

$$\cong \{ \tau \in K\sigma \mid \tau^{-1}y \in B \}$$

$$\cong \{ \tau \in K \mid \tau y^1 \in B \} ,$$

where $y^1 = \sigma^{-1}y$.

5.2. Define a map

$$\psi: \{ \tau \in K \mid \tau y^1 \in B \} \to Ky^1 \cap B$$

by $\psi(\tau) = \tau y^1$. Then $\psi$ is surjective and proper. Using lemma 1 on $C(y, \varnothing) = Ky^1$ and $y = y^1$, we find that $Ky^1 \cap B$ has the homotopy type of a point.

5.3. Again by the Leray spectral sequences, it is enough to show, that the fibers of $\psi$ have the homotopy type of a point. If $z \in Ky^1 \cap B$, then

$$\psi^{-1}(z) = \{ \tau \in K \mid \tau y^1 = z \}$$

$$= K \cap \{ \sigma \in G \mid \sigma y^1 = z \}$$

$$= K \cap \sigma^1G_{y^1} ,$$

where $\sigma^1 y^1 = z$.

This fiber is homotopy equivalent to a point according to the choice of $K$ in 3.3 having property 1).

5.4. Proof of Lemma 2. The mapping $\varphi$ of the pair

$$(\sigma \times A \cap \varphi^{-1}(B), K\sigma \times A \cap \varphi^{-1}(B))$$

onto the pair $(\sigma A \cap B, K\sigma A \cap B)$ gives a series of homomorphisms between the cohomology sequences. Now two of each three consecutive homomorphisms are isomorphisms according to 4.2 and 5.1. Hence the five-lemma can be applied to the remaining homomorphisms.

6. Real and complex index of functions.

6.1. Let $M$ be a complex $n$-dimensional manifold and $f: M \to \mathbb{R}$ a $C^2$-function. Then for any $p \in M$ and coordinate system $z_j = x_j + ix_{n+j}$ $j = 1, \ldots, n$, around $p$, the quadratic Levi form is

$$L_f(p, w) = \sum_{i,k} \frac{\partial^2 f(p)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_k, \quad w \in \mathbb{C}^n .$$
This form is known to be independent of the coordinates chosen and to be real valued. So we can define the complex index \( \text{Index}_C(f,p) \) as the maximum dimension of a complex subspace of \( \mathbb{C}^n \) on which \( L_j(p) \) is negative definite.

6.2. If we consider \( M \) as a real \( 2n \)-dimensional manifold with coordinates \( x_j, j = 1, \ldots, 2n \), we have the quadratic Hessian

\[
H_f(p,v) = \sum_{i,k} \frac{\partial^2 f(p)}{\partial x_k \partial x_j} v_k v_j, \quad v \in \mathbb{R}^{2n}.
\]

This form is known to be independent of the coordinates chosen, when \( df(p) = 0 \). So we can define the real index \( \text{Index}_R(f,p) \) as the maximum dimension of a real subspace of \( \mathbb{R}^{2n} \) on which \( H_f(p) \) is negative definite.

6.3. Lemma 4. \( \text{Index}_R(f,p) \geq \text{Index}_C(f,p) \).

Proof. Define

\[
E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Let \( w_j = v_j + i v_{n+j} \) and \( z_j = x_j + i x_{n+j} \). Then the formula

\[
L_f(p,w) = \frac{1}{4} (H_f(p,v) + (E^{-1} H_f E)(p,v))
\]

holds for all \( p \in M \). If we compute dimensions of the subspaces where the forms are positively semi-definite, we see, that

\[
2n - 2 \text{Index}_C(f,p) \geq 2(2n - \text{Index}_R(f,p)) - 2n
\]

and this proves the lemma.


Define \( M(\varrho) = K(\varrho) \sigma A \cap A \) for \( \varrho \leq r \).

There exists a function \( f: M(r) \to \mathbb{R} \) such that the Levi form of \( f \) has at least \( s \) negative eigenvalues, and for all \( \varrho, 0 \leq \varrho \leq r \),

\[
M(\varrho) = \{ x \in M(r) \mid f(x) \leq \alpha(\varrho) \},
\]

[1, Satz 1]. Further for any \( \varrho_0 \), \( 0 < \varrho_0 < r \), there exists a \( k > 0 \), such that

1) The Levi form of \( -e^{-kt} \) has at least \( s + 1 \) negative eigenvalues in \( M(r) \setminus M(\varrho_0) \),

2) For all \( \varrho, 0 \leq \varrho \leq r \),

\[
M(\varrho) = \{ x \in M(r) \mid -e^{-k f(x)} \leq -e^{-k \alpha(\varrho)} \}.
\]
Let $\varepsilon > 0$ be small enough. From [5, lemma 22.4, p. 119] we find $g_\varepsilon$ approximating $f$ on the set
\[ \{ x \in M(r) \mid g_0 + \varepsilon \leq f(x) \leq \alpha(\frac{1}{2}r) + \varepsilon \} \]
so well, that for all $x$ we have $|f(x) - g_\varepsilon(x)| < \varepsilon$ and
\[ \text{Index}_R(g_\varepsilon, x) = \text{Index}_R(f, x). \]

Lemma 4 says, that this index is at least $s + 1$. Define
\[ K_0(\varepsilon) = \{ x \in M(r) \mid g_\varepsilon \leq \alpha(g_0) + \varepsilon \} \]
\[ K(\varepsilon) = \{ x \in M(r) \mid g_\varepsilon(x) \leq \alpha(\frac{1}{2}r) + \varepsilon \}. \]

Then from [5, Theorem 3.2, p. 14] $H^m(K(\varepsilon), K_0(\varepsilon); Z) = 0$ for $0 \leq m \leq s$. Now
\[ \bigcap_{\varepsilon > 0} K(\varepsilon) = M(\frac{1}{2}r) \quad \text{and} \quad \bigcap_{\varepsilon > 0} K_0(\varepsilon) = M(g_0), \]
so when $\varepsilon \to 0$, we get $H^m(M(\frac{1}{2}r), M(g_0); Z) = 0$ for $0 \leq m \leq s$. Finally letting $g_0 \to 0$ we get $\bigcap_{g_0 > 0} M(g_0) = M(0)$, and hence
\[ H^m(M(\frac{1}{2}r), M(0); Z) = 0 \]
for $0 \leq m \leq s$, that is
\[ H^m(B \sigma A \cap A, \sigma A \cap A; Z) = 0 \]
for $0 \leq m \leq s$.

8. Proof of the theorems.

8.1. From lemmata 2\textsuperscript{1} and 3\textsuperscript{1} the homomorphisms $j^q$ and $k^q$ are isomorphisms for $0 \leq q \leq s$
\[ j^q: H^q(K \sigma \times \hat{A} \cap \hat{p}^{-1}(B); Z) \to H^q(\{ \sigma \} \times \hat{A} \cap \hat{p}^{-1}(B); Z), \]
\[ k^q: H^q(K \sigma \times \hat{A} \cap \hat{p}^{-1}(\hat{A} \cap B); Z) \to H^q(\{ \sigma \} \times \hat{A} \cap \hat{p}^{-1}(\hat{A} \cap B); Z), \]
and they are injective for $q = s + 1$.

Let $p: G \times \hat{A} \to G$ be the projection on $G$, and $p^1$ and $p^{11}$ the restrictions of $p$ to $\hat{p}^{-1}(B)$ and $\hat{p}^{-1}(\hat{A} \cap B)$ respectively, both mapping onto $G$. Because $j^q$ and $k^q$ are isomorphisms, the sheafs $R^q p^1 \ast Z$ and $R^q p^{11} \ast Z$ are locally constant for $q \leq s$, and because $\tau_1(G) = 0$, they are constant for $q \leq s$.

Further the maps
\[ H^0(G, R^{s+1} p^1 \ast Z) \to H^{s+1}(\hat{A} \cap \hat{B}; Z), \]
\[ H^0(G, R^{s+1} p^{11} \ast Z) \to H^{s+1}(\hat{A} \cap \hat{B}; Z) \]
are injective, because \( j^{s+1} \) and \( k^{s+1} \) are injective. So in the following commutative diagram defined by the inclusion \( \hat{\varphi}^{-1}(\hat{A} \cap \hat{B}) \subseteq \hat{\varphi}^{-1}(\hat{B}) \),

\[
H^0(G, R^{s+1}p^1_\ast Z) \to H^0(G, R^{s+1}p^{11}_\ast Z) \\
H^{s+1}(\hat{A} \cap \hat{B}; Z) \cong H^{s+1}(\hat{A} \cap \hat{B}; Z)
\]

both vertical maps are injective, hence also the upper map must be injective.

8.2. **Lemma 5.** If the inclusion \( \hat{\varphi}^{-1}(\hat{A} \cap \hat{B}) \subseteq \hat{\varphi}^{-1}(\hat{B}) \) induces isomorphisms

\[
H^j(\hat{\varphi}^{-1}(\hat{B}); Z) \cong H^j(\hat{\varphi}^{-1}(\hat{A} \cap \hat{B}); Z)
\]

for \( j < i \)

and \( H_j(\hat{A} \cap \hat{B}; Z) = 0 \) for \( 1 \leq j \leq i \), then the inclusion induces an isomorphism for \( j = i \leq s \), and a monomorphism for \( j = i = s + 1 \).

**Proof.** Consider the Leray spectral sequences for \( p^1 \) and \( p^{11} \) with mappings induced by inclusion

\[
H^i(\hat{\varphi}^{-1}(\hat{B}), Z) \leftarrow E_2^{r \ast} = H^r(G, R^q p^1_\ast Z) \\
H^i(\hat{\varphi}^{-1}(\hat{A} \cap \hat{B}), Z) \leftarrow \tilde{E}_2^{r \ast} = H^r(G, R^q p^{11}_\ast Z).
\]

From 8.1. follows, that we have the following commutative diagram

\[
E_2^{r \ast} \to H^r(G, H^q(\hat{A} \cap \hat{B}; Z)) \\
\cong
\hat{E}_2^{r \ast} \to H^r(G, H^q(\hat{A} \cap \hat{B}; Z))
\]

with isomorphisms for \( q \leq s \), and injective maps for \( q = s + 1 \) and \( r = 0 \).

We have exact rows in the commutative diagram

\[
0 \to E_2^{i,0} \to H^i(\hat{\varphi}^{-1}(\hat{B}); Z) \to E_2^{0,i} \to E_2^{i+1,0} \\
\cong
\hat{E}_2^{i,0} \to H^i(\hat{\varphi}^{-1}(\hat{A} \cap \hat{B}); Z) \to \hat{E}_2^{0,i} \to \hat{E}_2^{i+1,0}
\]

and three maps are isomorphisms for \( i \leq s \), hence so is the fourth. For \( i = s + 1 \), one of the three maps is injective only, but then the fourth map is injective too.

**Remark.** If lemma 5 is stated without \( \hat{\varphi} \) the proof is still valid.

8.3. **Lemma 6.** If the inclusion \( \hat{\varphi}^{-1}(\hat{A} \cap \hat{B}) \subseteq \hat{\varphi}^{-1}(\hat{B}) \) induces isomorphisms

\[
H^j(\hat{\varphi}^{-1}(\hat{B}), Z) \cong H^j(\hat{\varphi}^{-1}(\hat{A} \cap \hat{B}), Z)
\]

for \( j \leq i \)
and $H^i(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}; Z) = 0$ for $1 \leq j < i$, and $H^0(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}; Z) = Z$, for some $i \leq s + 1$, then $H^i(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}; Z) = 0$. If only 

$$H^i(\hat{\varphi}^{-1}(\hat{\mathcal{B}}); Z) \rightarrow H^i(\hat{\varphi}^{-1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}); Z)$$

is injective, we get only an injective map

$$H^i(\hat{\mathcal{B}}; Z) \rightarrow H^i(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}; Z)$$

induced by the inclusion $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}$.

**Proof.** We have $\hat{\varphi}: \hat{\mathcal{B}} \rightarrow S^{2n+1}$. Let $\hat{\varphi}': \hat{\varphi}^{-1}(\hat{\mathcal{B}}) \rightarrow \hat{\mathcal{B}}$ and $\hat{\varphi}'' : \hat{\varphi}^{-1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}) \rightarrow \hat{\mathcal{A}} \cap \hat{\mathcal{B}}$ be the restrictions of $\hat{\varphi}$. The spectral sequences for $\hat{\varphi}'$ and $\hat{\varphi}''$ are

$$H^i(\hat{\varphi}^{-1}(\hat{\mathcal{B}}); Z) \Leftarrow E^{ra}_2 = H^r(\hat{\mathcal{B}}, R^q\hat{\varphi}'_* Z) \Downarrow \cong$$

$$H^i(\hat{\varphi}^{-1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}); Z) \Leftarrow E^{ra}_2 = H^r(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, R^q\hat{\varphi}''_* Z) \Downarrow \cong$$

Because $\hat{\varphi}'$ and $\hat{\varphi}''$ are fiber bundles with the same fiber, $F$, we have the following commutative diagram

$$E^{ra}_2 \cong H^r(\hat{\mathcal{B}}, H^q(F, Z))$$

$$\Downarrow$$

$$E^{ra}_2 \cong H^r(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, H^q(F, Z)) .$$

For $i = 1$ we get a commutative diagram of exact sequences

$$0 \rightarrow H^1(\hat{\mathcal{B}}, H^0(F, Z)) \rightarrow H^1(\hat{\varphi}^{-1}(\hat{\mathcal{B}}), Z) \rightarrow H^1(F, Z)$$

$$0 \rightarrow H^1(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, H^0(F, Z)) \rightarrow H^1(\hat{\varphi}^{-1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}), Z) \rightarrow H^1(F, Z) ,$$

so the 5-lemma and $H^1(\hat{\mathcal{B}}, Z) = 0$ from [4, Proposition] give $H^1(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, Z) = 0$.

For $i > 1$ we get another commutative diagram of exact sequences

$$H^i(\hat{\mathcal{B}}, H^0(F, Z)) \rightarrow H^i(\hat{\varphi}^{-1}(\hat{\mathcal{B}}), Z) \rightarrow H^i(F, Z)$$

$$H^i(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, H^0(F, Z)) \rightarrow H^i(\hat{\varphi}^{-1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}), Z) \rightarrow H^i(F, Z) ,$$

so the 5-lemma and $H^i(\hat{\mathcal{B}}, Z) = 0$ from [4, Proposition] give $H^i(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, Z) = 0$ for $i \leq s$. If $i = s + 1$ and the middle vertical map is injective, we get induced an injective map

$$H^{s+1}(\hat{\mathcal{B}}, Z) \rightarrow H^{s+1}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}, Z) .$$
Remark. If lemma 6 is stated without ^ the proof is still valid.

8.4. Proposition. \( H^i(\tilde{B}, \tilde{A} \cap \tilde{B}; Z) = 0 \) for \( i \leq s + 1 \).

Proof. Induction using lemmata 5 and 6 gives \( H^i(\tilde{A} \cap \tilde{B}, Z) = 0 \) for \( 1 \leq i \leq s \) and

\[
H^{s+1}(\tilde{B}, Z) \rightarrow H^{s+1}(\tilde{A} \cap \tilde{B}, Z)
\]

injective. Because \( H^i(\tilde{B}, Z) = 0 \) for \( 1 \leq i \leq s \) by [4] and because \( H^0(\tilde{A} \cap \tilde{B}, Z) = Z \) by [2, Proposition 4], the proposition now follows.

8.5. Proof of Theorem 1. Induction using the remarks following lemmata 5 and 6 gives \( H^1(\tilde{A} \cap \tilde{B}; Z) = 0 \) and an isomorphism if \( s \geq 2 \)

\[
H^2(B, Z) \cong H^2(A \cap B, Z).
\]

By [2, Proposition 4], \( H^0(A \cap B, Z) = Z \) and by [4, Theorem] \( H^2(B, Z) = Z \). The exact sequence for the pair \( A \cap B \subseteq B \) is

\[
H^0(B, Z) \cong H^0(A \cap B, Z) \rightarrow H^1(B, A \cap B, Z) \rightarrow H^1(B, Z) \cong
\]

\[
\cong H^1(A \cap B, Z) \rightarrow H^2(B, A \cap B, Z) \rightarrow H^2(B, Z) \rightarrow H^2(A \cap B, Z)
\]

with isomorphisms \( \cong \) and the last map injective. Hence \( H^1(B, A \cap B, Z) = 0 \) and \( H^2(B, A \cap B, Z) = 0 \).

From the proposition and 2.5 follows, that \( H^i(B, A \cap B, Z) = 0 \) for \( 1 \leq i \leq s + 1 \), and the universal coefficient theorem then gives

\[
H_i(B, A \cap B, Z) = 0 \quad \text{for} \ 1 \leq i \leq s + 1.
\]

8.6. Proof of Theorem 2. Follows from 2.4, proposition and 2.3.

REFERENCES


