ON AN ASYMPTOTIC FORMULA OF RAMANUJAN

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1. Introduction.

Let \( \tau(n) \) denote the number of divisors of a positive integer \( n \). In 1915 S. Ramanujan (cf. [6], (3)) stated without proof the following asymptotic formula:

\[
(1.1) \quad \sum_{n \leq x} \tau^2(n) = Ax \log^2 x + Bx \log x + Cx \log x + Dx + O(x^{\frac{9}{4} + \epsilon}) ,
\]

for every \( \epsilon > 0 \), where \( A = \pi^2, B = (12\gamma - 3)\pi^2 - 36\pi^2\zeta'(2), \) etc., \( \gamma \) being Euler's constant, \( \zeta'(2) \) is the derivative of the Riemann Zeta function \( \zeta(s) \) at \( s = 2 \). He also stated that the order of the error term in (1.1) may be improved to \( O(x^{\frac{9}{4} + \epsilon}) \), on the assumption of the Riemann hypothesis. In 1922, B. M. Wilson [10] gave a proof of (1.1) with error term \( O(x^{\frac{9}{4} + \epsilon}) \) without assuming any hypothesis.

The object of the present paper is to further improve the order of the error term (denoted throughout the paper by \( E(x) \)) in (1.1).

Let \( \tau_d(n) \) denote the number of representations of \( n \) in the form \( n = d_1 d_2 d_3 d_4 \) and let \( \alpha \) denote the number which appears in the divisors problem for \( \tau_d(n) \), namely

\[
(1.2) \quad \sum_{n \leq x} \tau_d(n) = ax \log^2 x + bx \log x + cx \log x + dx + O(x^{\alpha}),
\]

where \( a = \frac{\gamma}{4}, b = 2\gamma - \frac{1}{4}, \) etc.

The formula (1.2) was originally obtained in 1881 by A. Piltz [5] with error term equal to \( O(x^{\frac{1}{4} \log^2 x}) \). In 1912, E. Landau [4] proved that \( \alpha = \frac{\gamma}{2} + \epsilon \) for every \( \epsilon > 0 \) and this result was improved further in 1922 by G. H. Hardy and J. E. Littlewood [2] to \( \alpha = \frac{1}{4} + \epsilon \). On the other hand, G. H. Hardy [1] in 1915 proved that \( \alpha \geq \frac{\gamma}{2} \). There is a conjecture (cf. [8, p. 270]) that \( \alpha = \frac{\gamma}{2} + \epsilon \). If this conjecture were true, then it would follow that \( \alpha < \frac{1}{2} \). For a discussion about the divisor problem for \( \tau_d(n) \), we refer to E. C. Titchmarsh (cf. [8, theorem 12.3 and theorem 12.6(B)]).

Throughout the paper we assume that the number \( \alpha \) appearing in (1.2) is strictly less than \( \frac{1}{2} \). With this assumption we prove in this paper that

\[
E(x) = O(x^{\frac{1}{4}} \exp\{-A \log^2 x (\log \log x)^{-\frac{1}{4}}\}),
\]

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where $A$ is a positive constant. Further, on the assumption of the Riemann hypothesis, we prove that

$$ E(x) = O(x^{(2-s)/(5-4s)}) \exp \{ A \log x (\log \log x)^{-1} \}, $$

where $A$ is a positive constant.

2. Preliminaries.

In this section, we prove some lemmas which are needed in our present discussion. Throughout the following $x$ denotes a real variable $\geq 3$. We need the following best known estimate concerning the Möbius function $\mu(n)$ obtained by A. Walfisz (cf. [9; Satz 3, p. 191]).

**Lemma 2.1.**

(2.1) \[ M(x) = \sum_{n \leq x} \mu(n) = O(x \delta(x)), \]

where

(2.2) \[ \delta(x) = \exp \{ -A \log^b x (\log \log x)^{-1} \}, \]

$A$ being a positive constant.

**Lemma 2.2.** For $s > 1$ and $r \geq 0$,

(2.3) \[ \sum_{n \leq x} n^{-s} \mu(n) \log^r n = (-1)^r \eta^{(r)}(s) + O(x^{-(s-1)} \delta(x) \log^r x), \]

where $\eta^{(0)}(s) = \eta(s) = \zeta(s)^{-1}$ and $\eta^{(r)}(s)$ for $r \geq 1$ denotes the $r$th derivative of $\eta(s) = \zeta(s)^{-1}$.

**Proof.** From the well-known formula (cf. [3, theorem 287]),

$$ \sum_{n=1}^{\infty} n^{-s} \mu(n) = \zeta(s)^{-1} = \eta(s), $$

we have

$$ \sum_{n=1}^{\infty} n^{-s} \mu(n) \log^r n = (-1)^r \eta^{(r)}(s), $$

so that

$$ \sum_{n \leq x} n^{-s} \mu(n) \log^r n = (-1)^r \eta^{(r)}(s) - \sum_{n > x} n^{-s} \mu(n) \log^r n. $$

Putting $f(n) = n^{-s} \log^r n$, it can be easily shown that

$$ f(n+1) - f(n) = O(n^{-(s+1)} \log^r n). $$
Therefore by partial summation and (2.1),
\[ \sum_{n>x} \mu(n)f(n) = -M(x)f([x] + 1) - \sum_{n>x} M(n)\{f(n + 1) - f(n)\} \]
\[ = O(x^{-(s-1)}\delta(x)\log^r x) + O(\sum_{n>x} n^{-s}\delta(n)\log^r n) \]
\[ = O(x^{-(s-1)}\delta(x)\log^r x) + O(\delta(x) \sum_{n>x} n^{-s}\log^r n) \]
\[ = O(x^{-(s-1)}\delta(x)\log^r x) + O(x^{-(s-1)}\delta(x)\log^r x) \]
\[ = O(x^{-(s-1)}\delta(x)\log^r x). \]

Hence the lemma follows.

**Lemma 2.3.** (Cf. [8, theorem 14–26(A), p. 316]). If the Riemann hypothesis is true, then
\[ (2.4) \quad M(x) = \sum_{n \leq x} \mu(n) = O(x^4 \omega(x)), \]
where
\[ (2.5) \quad \omega(x) = \exp\{A \log x (\log \log x)^{-1}\}, \]
A being a positive constant.

**Lemma 2.4.** If the Riemann hypothesis is true, then for \( s > 1, \)
\[ (2.6) \quad \sum_{n \leq x} n^{-s} \mu(n) \log^r n = (-1)^r \eta^{(r)}(s) + O(x^{4-s} \omega(x) \log^r x). \]

**Proof.** Following the same argument adopted in lemma 2.2, we get this lemma by making use of (2.4) instead of (2.1). In fact, we have only to replace \( \delta(x) \) in lemma 2.2 by \( x^{-4} \omega(x) \).

3. Main results.

In this section, we first prove a lemma and then prove the results mentioned in the introduction.

**Lemma 3.1.** \( \tau^2(n) = \sum_{d \mid n} \mu(d)\tau_d(n/d^2). \)

**Proof.** Since \( \mu(n) \) and \( \tau_d(n) \) are multiplicative it follows (cf. [7, lemma 2.4]) that the function on the right is a multiplicative function of \( n \). Since \( \tau^2(n) \) is also multiplicative, it is enough, if we verify the identity for \( n = p^a, \) where \( p \) is a prime and \( a \geq 1. \) We note that
\[ \tau_d(p^a) = (a + 1)(a + 2)(a + 3)/6 \]
(cf. [8, (1.2.6), p. 5]). We have
\[ \sum_{d \mid p} \mu(d)\tau_d(p/d^2) = \mu(1)\tau_d(p) = 4, \]
and for \( a \geq 2 \),
\[
\sum_{d|n \in \mathbb{P}} \mu(d) \tau_4(d^a/d^2) = \mu(1) \tau_4(p^a) + \mu(p) \tau_4(p^{a-2}) = (a+1)(a+2)(a+3)/6 - (a-1)a(a+1)/6 = (a+1)^2.
\]

Hence the lemma follows.

**Theorem 3.1.** For \( x \geq 3 \),
\[
(3.1) \quad \sum_{n \leq x} \tau^2(n) = \frac{ax \log^3 x}{\zeta(2)} + \frac{b}{\zeta(2)} + 6ax \eta^{(1)}(2) x \log^2 x + \frac{c}{\zeta(2)} + 4b \eta^{(1)}(2) + 12ax \eta^{(2)}(2) x \log x + \frac{d}{\zeta(2)} + 2c \eta^{(1)}(2) + 4b \eta^{(2)}(2) + 8ax \eta^{(3)}(2) x + E(x),
\]
where \( E(x) = O(x^4 \delta(x)) \), \( \delta(x) \) being given by (2.2), \( a, b, c, d \) are constants in the asymptotic formula (1.2) and \( \eta^{(r)}(s) \) is the \( r \)th derivative of \( \eta(s) = \zeta(s)^{-1} \) at \( s = 2 \) for \( r = 1, 2, 3 \).

**Proof.** In virtue of lemma 3.1 above, we have
\[
(3.2) \quad \sum_{n \leq x} \tau^2(n) = \sum_{n \leq x} \sum_{d|n} \mu(d) \tau_4(d) = \sum_{d|n \leq x} \mu(d) \tau_4(d),
\]
the summation being extended over all ordered pairs \((d, \delta)\) such that \( d^2 \delta \leq x \).

Let \( z = x^4 \). Further, let \( 0 < \varrho = \varrho(x) < 1 \), where the function \( \varrho \) will be suitably chosen later. From (3.2), we have
\[
\sum_{n \leq x} \tau^2(n) = \sum_{n \varrho \leq x} \mu(n) \tau_4(r).
\]

If \( n^2 r \leq x \), then both \( n \geq \varrho \) and \( r \geq \varrho^{-2} \) can not simultaneously hold good and so we have
\[
\sum_{n \leq x} \tau^2(n) = \sum_{n \varrho \leq x} \mu(n) \tau_4(r) + \sum_{n \varrho \leq x} \mu(n) \tau_4(r) - \sum_{n \varrho \leq x} \mu(n) \tau_4(r)
\]
\[
= S_1 + S_2 - S_3,
\]
say. Now, by (1.2),
\[
S_1 = \sum_{n \varrho \leq x} \mu(n) \tau_4(r) = \sum_{n \varrho \leq x} \mu(n) \sum_{r \leq x \varrho - 2} \tau_4(r)
\]
\[
= \sum_{n \varrho \leq x} \mu(n) \{a x n^{-2} \log^3 (x n^{-2}) + b x n^{-2} \log^2 (x n^{-2}) + c x n^{-2} \log (x n^{-2}) + d x n^{-2} + O((x n^{-2})^a)\}
\]
\[
= (a x \log^3 x + b x \log^2 x + c x \log x + d x) \sum_{n \varrho \leq x} n^{-2} \mu(n) - 2x (3a \log^2 x + 2b \log x + c) \sum_{n \varrho \leq x} n^{-2} \mu(n) \log n + 4x (3a \log x + b) \sum_{n \varrho \leq x} n^{-2} \mu(n) \log^2 n - 8ax \sum_{n \varrho \leq x} n^{-2} \mu(n) \log^3 n \}
+ O(x^a \sum_{n \varrho \leq x} n^{-2}).
Since \(0 < 2\alpha < 1\), by our assumption, we have

\[
x^\alpha \sum_{n \leq \mathfrak{e}} n^{-2\alpha} = O(x^\alpha (\mathfrak{e}z)^{1-2\alpha}) = O(e^{1-2\alpha z}).
\]

Hence applying lemma 2.2 for \(r = 0, 1, 2, 3\) and \(s = 2\), we get that

\[
S_1 = (ax \log^2 x + bx \log^2 x + cx \log x + dx)\{\zeta(2)^{-1} + O(\delta(\mathfrak{e}z)/\mathfrak{e}z)\} - 2x(3a \log x + 2b \log x + c)\{-\eta^{(1)}(2) + O(\delta(\mathfrak{e}z) \log (\mathfrak{e}z)/\mathfrak{e}z)\} + 4x(3a \log x + b)\{\eta^{(2)}(2) + O(\delta(\mathfrak{e}z) \log^2 (\mathfrak{e}z)/\mathfrak{e}z)\} - 8ax\{-\eta^{(3)}(2) + O(\delta(\mathfrak{e}z) \log^3 (\mathfrak{e}z)/\mathfrak{e}z)\} + O(e^{1-2\alpha z}).
\]

(3.4)

\[
= ax \log^3 x/\zeta(2) + (b/\zeta(2) + 6a \eta^{(1)}(2)) x \log^2 x + (c/\zeta(2) + 4b \eta^{(1)}(2) + 12a \eta^{(2)}(2)) x \log x + (d/\zeta(2) + 2c \eta^{(1)}(2) + 4b \eta^{(2)}(2) + 8a \eta^{(3)}(2)) x + O(e^{-1}z \delta(\mathfrak{e}z) \log^3 z) + O(e^{1-2\alpha z}).
\]

We have

\[
S_2 = \sum_{\substack{n \leq z \leq x \quad \tau_4(n)}} \mu(n) \tau_4(n) = \sum_{r \leq e^{-\mathfrak{e}}} \tau_4(r) \sum_{n \leq r^{-1}} \mu(n) = \sum_{r \leq e^{-\mathfrak{e}}} \tau_4(r) M((x/r)^i)
\]

\[
= O(x^i \sum_{r \leq e^{-\mathfrak{e}}} \tau_4(r) r^{-i} \delta((x/r)^i)) ,
\]

by (2.1). Since \(\delta(x)\) is monotonic decreasing and \((x/r)^i > \mathfrak{e}z\), we have \(\delta((x/r)^i) \leq \delta(\mathfrak{e}z)\). Also, by (1.2),

\[
\sum_{r \leq e^{-\mathfrak{e}}} \tau_4(r) r^{-i} = O(e^{-1} \log^3 (e^{-\mathfrak{e}})).
\]

Hence

(3.5)

\[
S_2 = O(e^{-1}z \delta(\mathfrak{e}z) \log^3 (1/\mathfrak{e})).
\]

Also, we have by (2.1) and (1.2),

\[
S_3 = \sum_{\substack{n \leq \mathfrak{e}z \quad \tau_4(n)}} \mu(n) \tau_4(n) = \sum_{r \leq e^{-2}} \tau_4(r) M(\mathfrak{e}z)
\]

\[
= O(e^{-2} \log^3 (e^{-2}) \mathfrak{e}z \delta(\mathfrak{e}z))
\]

(3.6)

\[
= O(e^{-1}z \delta(\mathfrak{e}z) \log^3 (e^{-1})).
\]

Hence by (3.3), (3.4), (3.5) and (3.6), we have

(3.7)

\[
\sum_{n \leq x} \tau^2(n) = ax \log^3 x/\zeta(2) + (b/\zeta(2) + 6a \eta^{(1)}(2)) x \log^2 x + (c/\zeta(2) + 4b \eta^{(1)}(2) + 12a \eta^{(2)}(2)) x \log x + (d/\zeta(2) + 2c \eta^{(1)}(2) + 4b \eta^{(2)}(2) + 8a \eta^{(3)}(2)) x + O(e^{-1}z \delta(\mathfrak{e}z) \log^3 z) + O(e^{-1}z \delta(\mathfrak{e}z) \log^3 (e^{-1})) + O(e^{1-2\alpha z}).
\]
Now, we choose
\[ e = \varrho(x) = \{\delta(x^\pm)\}^4, \]
and write
\[ f(x) = \log^2(x^\pm)\{\log\log(x^\pm)\}^{-1} = (\frac{1}{2})^2u^2(v - \log 4)^{-1}, \]
where \( u = \log x \) and \( v = \log\log x \).

\[ f(x) = \log^2(x^\pm)\{\log\log(x^\pm)\}^{-1} = (\frac{1}{2})^2u^2(v - \log 4)^{-1}, \]

For \( v \geq 2\log 4 \), that is, \( u \geq 16 \), \( x \geq e^{16} \), we have
\[ v^{-\frac{1}{2}} \leq (u - \log 4)^{-1} \leq (\frac{1}{2}v)^{-\frac{1}{2}}, \]
so that
\[ \frac{1}{2}(\frac{1}{2})^2u^2v^{-\frac{1}{2}} \leq f(x) \leq (\frac{1}{2})^2u^2v^{-\frac{1}{2}}. \]

We assume without loss of generality that the constant \( A \) in (2.2) is less than 1.

By (3.8), (2.2) and (3.9), we have
\[ \varrho = \exp\{-\frac{1}{4}Af(x)\}. \]

By (10.10), we have
\[ (\frac{1}{2})^2u^2v^{-\frac{1}{2}} \leq \frac{1}{4}u. \]

Hence by (3.11), (3.12), (3.13) and the above,
\[ \varrho \geq \exp\{-A(\frac{1}{2})^2u^2v^{-\frac{1}{2}}\} \geq \exp\{-\frac{1}{4}u\} \geq \exp\{-\frac{1}{4}\log x\}, \]
so that \( \varrho \geq x^{-\frac{1}{4}}. \) Hence
\[ \log(\varrho^{-1}) \leq \log(x^\pm) = O(\log x) \quad \text{and} \quad \varrho x \geq x^\pm. \]

Since \( \delta(x) \) is monotonic decreasing, \( \delta(\varrho x) \leq \delta(x^\pm) \), by (3.8) and so by (3.11) and (3.13), we have
\[ \varrho^{-1}\delta(\varrho x) \leq \varrho \leq \exp\{-\frac{1}{4}A(\frac{1}{2})^2u^2v^{-\frac{1}{2}}\}. \]

Hence by (3.14) and (3.15), the first and second \( O \)-terms of (3.7) are each equal to
\[ O(x^\pm\exp\{-\frac{1}{4}A(\frac{1}{2})^2u^2v^{-\frac{1}{2}}\}\log^3x) \]
which is
\[ O(x^\pm\exp\{-\frac{1}{4}A(1 - 2x)(\frac{1}{2})^2u^2v^{-\frac{1}{2}}\}) \],
since \( 0 < 1 - 2x < 1 \), by our assumption.

By (3.13) and (3.11), we see that the third \( O \)-term in (3.7) is also of the above order. Thus, if \( E(x) \) denotes the sum of the three error terms in (3.7), we have
\[ E(x) = O(x^\pm\exp\{-B\log^3x(\log\log x)^{-1}\}), \]
where \( B \) is a positive constant.
Theorem 3.2. If the Riemann hypothesis is true, then the error term $E(x)$ in the asymptotic formula for $\sum_{n \leq x} \tau^2(n)$ is

$$O(x^{(2-\alpha)/(5-4\alpha)} \omega(x)),$$

where $\alpha$ is the number given by (1.2) and $\omega(x)$ is given by (2.5).

Proof. Following the same procedure adopted in theorem 3.1 and making use of lemma 2.4 for $r=0, 1, 2, 3$ and $s=2$ instead of lemma 2.2 for $r=0, 1, 2, 3$ and $s=2$, we get that

$$(3.17) \quad E(x) = O(x^{-\frac{3}{2}} \log x \omega(x)) + O(x^{-\frac{1}{2}} \log x \omega(x)) + O(x^{1-2\alpha}).$$

Now, choosing $\theta = z^{-(5-4\alpha)-1}$, we see that $0 < \theta < 1$, $\theta^{-1} < z$, so that $\log(\theta^{-1}) < \log z$ and

$$\theta^{-\frac{3}{2}} = \theta^{1-2\alpha} = x^{(2-\alpha)/(5-4\alpha)}.$$

Since $\omega(x)$ is monotonic increasing and $\theta z < z$, we have $\omega(\theta z) < \omega(z)$. Hence by (3.17) and the above, we have

$$E(x) = O(x^{(2-\alpha)/(5-4\alpha)} \omega(x) \log x) = O(x^{(2-\alpha)/(5-4\alpha)} \omega(x)).$$

Hence theorem 3.2 follows.

References


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