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A SIMPLIFIED CONSTRUCTIVE PROOF OF THE EXISTENCE AND UNIQUENESS OF HAAR MEASURE

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More than 20 years have passed since H. Cartan gave his constructive proof [1] avoiding the axiom of choice and proving existence and uniqueness simultaneously. In spite of this, his proof has not been generally adopted in subsequent presentations of the subject. It is considered to be more complicated and less intuitive than the traditional proofs, going back to A. Haar [2] and A. Weil [6] (Cf. e.g. [4,p. 113]). The aim of this paper is to present a version of the constructive proof which is as simple and intuitive as the traditional non-constructive proofs, if not equally short.

1. Preliminaries and notations.

In the sequel, G shall be an arbitrary, but fixed, locally compact group, and L shall be the class of continuous real valued functions with compact support on G. For every member V of the neighbourhood filter $\mathscr{V}$ of the identity e, the symbol L_V denotes the class of all $f \in L$ vanishing off V. For every $f \in L$ the *conjugate function* f^* is defined by $f^*(x)= f(x^{-1})$. For every $f \in L$ and $s \in G$, the *left* and *right translates* f_s and f^s, are defined by $f_s(x)=f(s^{-1}x)$ and $f^s(x)=f(xs)$. (These are the conventions of [5]).

It follows from the local compactness, that if $f, \varphi \in L^+$ and $\varphi \neq 0$, then there exist elements $s_1,\ldots,s_n \in G$ and positive numbers $\alpha_1,\ldots,\alpha_n$ such that

$$(1.1) \qquad f \leqq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} .$$

Thus we may define

$$(1.2) \qquad (\overline{f:\varphi}) = \inf\left\{ \sum_{i=1}^{n} \alpha_i \mid f \leqq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \right\},$$

$$(1.3) \qquad (\underline{f:\varphi}) = \sup\left\{ \sum_{j=1}^{m} \beta_j \mid \sum_{j=1}^{m} \beta_j \varphi_{t_j} \leqq f \right\}.$$

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It should be noted that for every $f \in L^+$, $f \neq 0$, one has $(\overline{f:\varphi}) > 0$, and there is a $V \in \mathscr{V}$ such that $(\underline{f:\varphi}) > 0$ whenever $\varphi \in L_V^+$.

We list the following standard properties for later references:

$$f \leqq g \quad \Rightarrow \quad (\overline{f:\varphi}) \leqq (\overline{g:\varphi}) , \tag{1.4}$$

$$(\overline{f:\varphi}) \leqq (\overline{f:\psi})(\overline{\psi:\varphi}) , \tag{1.5}$$

$$(\overline{f_s:\varphi}) = (\overline{f:\varphi}) , \tag{1.6}$$

$$(\overline{\alpha f:\varphi}) = \alpha(\overline{f:\varphi}) , \tag{1.7}$$

$$(\overline{\textstyle\sum_{i=1}^n f_i:\varphi}) \leqq \textstyle\sum_{i=1}^n (\overline{f_i:\varphi}) . \tag{1.8}$$

The dual statements are valid for the function $(f,\varphi) \to (\underline{f:\varphi})$.

PROPOSITION 1.1. *Let $f_1,\ldots,f_n \in L$ and $\lambda > 1$ be arbitrary, and write $f = \sum_{i=1}^n f_i$. Then there exists a $V \in \mathscr{V}$ such that for every $\varphi \in L_V^+$, $\varphi \neq 0$,*

$$\sum_{i=1}^{n} (\overline{f_i:\varphi}) \leqq \lambda(\overline{f:\varphi}) . \tag{1.9}$$

PROOF. Cf. for example [4, p. 114].

PROPOSITION 1.2. *Let $\{f_i\}$ be some generalized sequence on L^+ which converges uniformly to $f \neq 0$, and for which all f_i vanish off some fixed compact K. Then*

$$\lim_i \frac{(\overline{f_i:\varphi})}{(\overline{f:\varphi})} = 1 , \tag{1.10}$$

uniformly in φ.

PROOF. Let $f_0 \in L^+$ be chosen such that $f_0(x) = 1$ for $x \in K$. Let $\varepsilon > 0$ be arbitrary and write $\delta = (\overline{f_0:f})^{-1}\varepsilon$. By assumption there exists an index j such that for all $i > j$:

$$|f - f_i| < \delta f_0 .$$

Writing this relation as two inequalities, using (1.4), (1.7) and (1.8), one obtains

$$|(\overline{f:\varphi}) - (\overline{f_i:\varphi})| < \delta(\overline{f_0:\varphi}) ,$$

for all $\varphi \in L^+$, $\varphi \neq 0$. Dividing through by $(\overline{f:\varphi})$ and making use of (1.5), one obtains

$$\left| 1 - \frac{(\overline{f_i:\varphi})}{(\overline{f:\varphi})} \right| < \varepsilon$$

for all $i > j$ and $\varphi \in L^+$, $\varphi \neq 0$.

Corollary. *If $\{f_i\}$ is a generalized sequence on L^+ and $f_i \uparrow f \in L^+$, $f \neq 0$, then*

$$(1.11) \qquad \frac{\overline{(f_i:\varphi)}}{\overline{(f:\varphi)}} \uparrow 1 ,$$

and the convergence is uniform in φ.

Proof. Application of Dini's lemma.

Proposition 1.3. *If $g, \psi \in L^+$ and $\psi \neq 0$, then*

$$(1.12) \qquad \underline{(g:\psi)} \leqq \overline{(g:\psi)} .$$

Proof. Let $\lambda > 1$ be arbitrary, and determine $t_1,\ldots,t_m \in G$ and $\beta_1,\ldots,\beta_m > 0$ such that

$$(1.13) \qquad \sum_{j=1}^{m} \beta_j \psi_{t_j} \leqq g, \qquad \underline{(g:\psi)} \leqq \lambda \sum_{j=1}^{m} \beta_j .$$

Choose $V \in \mathscr{V}$ such that (1.9) holds with $\beta_j \psi_{t_j}$ in the place of f_i. By (1.4)–(1.7) and (1.13), the following relations hold for any $\varphi \in L_V^+$:

$$\sum_{j=1}^{m} \beta_j \overline{(\psi:\varphi)} \leqq \lambda \overline{(g:\varphi)} \leqq \lambda \overline{(g:\psi)}\,\overline{(\psi:\varphi)} .$$

By (1.13), this implies

$$\underline{(g:\psi)} \leqq \lambda^2 \overline{(g:\psi)} ,$$

and the proof is complete.

2. Reduction to a separation theorem.

A functional I on L^+ will be said to be *admissible* if it is non-trivial (i.e. $\neq 0$), isotone (or order-preserving), positively homogeneous, additive, and *left-invariant* in the sense that

$$(2.1) \qquad I(f) = I(f_s) ,$$

for all $f \in L^+$ and all $s \in G$. It follows from (1.1) that for every admissible functional I:

$$(2.2) \qquad \varphi \in L^+,\ \varphi \neq 0 \quad \Rightarrow \quad I(\varphi) > 0 .$$

To every admissible functional I on L^+ one may associate a *pre-ordering* (reflexive and transitive relation), defined by

$$(2.3) \qquad f \leqq g \ (\mathrm{mod}\, I) \quad \Leftrightarrow \quad I(f) \leqq I(g) .$$

The existence and uniqueness of Haar measure will follow from the existence and uniqueness, up to a positive factor, of an admissible functional on L^+. In this connection we observe that an admissible functional is determined up to a positive factor by its associated pre-ordering. In fact, one has the following, somewhat stronger, statement:

PROPOSITION 2.1. *Two admissible functionals with comparable (finer-coarser) pre-orderings differ only by some positive factor.*

PROOF. Let I, J be two admissible functionals on L^+ such that

$$I(g) \leqq I(h) \quad \Rightarrow \quad J(g) \leqq J(h) . \tag{2.4}$$

Let φ be an arbitrary, but fixed, non-zero member of L^+. By (2.2) there is a positive number α such that

$$I(\varphi) = \alpha J(\varphi) .$$

For an arbitrary $f \in L^+$ there exists a number $\beta \geqq 0$ such that

$$I(f) = \beta I(\varphi) .$$

Applying (2.4) twice with $g=f$, $h=\beta\varphi$, and $g=\beta\varphi$, $h=f$, one obtains

$$J(f) = \beta J(\varphi) .$$

Hence

$$I(f) = \beta I(\varphi) = \alpha\beta J(\varphi) = \alpha J(f) .$$

This accomplishes the proof.

Proposition 2.1 shows the importance of the pre-orderings associated with admissible functionals and proves it sufficient to find one such relation *comparable* with any other. In this connection it is natural to consider the following definition of "relative size" on L^+:

$$f \leqq g \quad \Leftrightarrow \quad \underline{(f:\varphi)} \leqq \overline{(g:\varphi)} \quad \text{for all} \quad \varphi \in L^+, \ \varphi \neq 0 . \tag{2.5}$$

This type of definition is of course, not new. In principle it is identical with Eudoxos' definition of relative size for incommensurable proportions. In the present context it gains importance by virtue of the following:

PROPOSITION 2.2. *The relation* (2.5) *is coarser than the pre-ordering associated with any admissible functional on* L^+.

PROOF. Let I be an admissible functional on L^+ and let $f, g \in L^+$, and $I(f) \leqq I(g)$. If $\varphi \in L^+$, $\varphi \neq 0$, and

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leqq f, \qquad g \leqq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} ,$$

then

$$I(\varphi)\sum_{j=1}^{m}\beta_j \leqq I(\varphi)\sum_{i=1}^{n}\alpha_i .$$

By (2.2),

$$\sum_{j=1}^{m}\beta_j \leqq \sum_{i=1}^{n}\alpha_i .$$

Hence $\underline{(f:\varphi)} \leqq \overline{(g:\varphi)}$, and the proposition is proved.

By Propositions 2.1–2.2, it suffices to prove that the relation (2.5) itself is a pre-ordering associated with some admissible functional. To this end we claim that for all non-zero $f, g \in L^+$

$$\inf_\varphi \frac{\overline{(g:\varphi)}}{\underline{(f:\varphi)}} = \lim_\varphi \frac{\overline{(g:\varphi)}}{\underline{(f:\varphi)}} = \lim_\varphi \frac{\overline{(g:\varphi)}}{\overline{(f:\varphi)}}, \tag{2.6}$$

where the indices $\varphi \in L^+$, $\varphi \neq 0$, are directed by inclusion of the sets $\{x \mid \varphi(x) > 0\}$. By (1.4)–(1.9), the limit at the right hand side of (2.6) would define an admissible functional for any fixed $f \in L^+$, $f \neq 0$, and by the alternative expression at the left hand side of (2.6), its associated pre-ordering would be exactly the one defined by (2.5). Hence it suffices to prove the claim (2.6).

The relevant information in this connection turns out to be the following separation property for functions in L_V^+:

(S). *If $f, g \in L^+$ and $f(x) < g(x)$ for $x \in \operatorname{spt}(f)$, then there is a $V \in \mathscr{V}$ such that every $\varphi \in L_V^+$ admits group elements $s_1, \ldots, s_n$ and positive numbers $\alpha_1, \ldots, \alpha_n$ such that*

$$f \leqq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \leqq g .$$

An immediate consequence of (S) is the following: If $f, g \in L^+$ and $f(x) < g(x)$ for $x \in \operatorname{spt}(f)$, then there is a $V \in \mathscr{V}$ such that for every $\varphi \in L_V^+$

$$\overline{(f:\varphi)} \leqq \underline{(g:\varphi)} . \tag{2.7}$$

Proposition 2.3. *The separation property* (S) *implies the claim* (2.6).

Proof. 1) We first prove that for every non-zero $f \in L^+$

$$\lim_\varphi \frac{\overline{(f:\varphi)}}{\underline{(f:\varphi)}} = 1 . \tag{2.8}$$

Let $\lambda > 1$ be arbitrary, and write $f_n = (f - 1/n)^+$ for $n = 1, 2, \ldots$, By the Corollary to Proposition 1.2, there is a natural number m such that for all $\varphi \in L^+$, $\varphi \neq 0$,

$$\lambda^{-1} \leqq \frac{\overline{(f_m:\varphi)}}{\overline{(f:\varphi)}} .$$

Now we apply the separation property (S) in the form (2.7) with f_m in the place of f and f in the place of g. Thus for some $V \in \mathscr{V}$ and all $\varphi \in L_V^+$

$$\overline{(f_m:\varphi)} \leqq \underline{(f:\varphi)} .$$

Combining the last two inequalities and applying Proposition 1.3, one obtains

$$1 \leqq \frac{\overline{(f:\varphi)}}{\underline{(f:\varphi)}} \leqq \lambda$$

for every $\varphi \in L_V^+$. This completes the proof of (2.8).

2) Now only the first equality of (2.6) remains to be verified, and we shall be through if we can prove that the generalized sequence $\{\overline{(g:\varphi)}/\underline{(f:\varphi)}\}_\varphi$ is 'nearly monotone'' in the following precise sense: For every $\psi \in L^+$, $\psi \neq 0$, and every $\lambda > 1$, there is a $V \in \mathscr{V}$ such that for every $\varphi \in L_V^+$

$$\frac{\overline{(g:\varphi)}}{\underline{(f:\varphi)}} \leqq \lambda \frac{\overline{(g:\psi)}}{\underline{(f:\psi)}} . \tag{2.9}$$

By (1.5) and its dual, if suffices to take a $V \in \mathscr{V}$ such that $\overline{(\psi:\varphi)}/\underline{(\psi:\varphi)} \leqq \lambda$ for all $\varphi \in L_V^+$. This is possible in virtue of (2.8), and the proof is complete.

Remark. Our motivation for introducing the generalized sequence $\{\overline{(g:\varphi)}/\underline{(f:\varphi)}\}_\varphi$ and not restricting ourselves to the study of $\{\overline{(g:\varphi)}/\overline{(f:\varphi)}\}_\varphi$ is the "near monotonity" of the former, yielding the equality (2.6), which is the clue to the uniqueness problem, in virtue of the two simple Propositions 2.1–2.2.

3. Proof of the separation theorem.

We first sketch the idea of the proof. To establish the property (S), we shall show that some suitably chosen function h between f and g, can be uniformly approximated by functions

$$\sum_{i=1}^{n} \alpha_i \varphi_{s_i} ,$$

where φ is "sufficiently concentrated" around e.

Such an approximation could be easily obtained by means of the Haar integral. The theorem on (right) approximate identities in the convolution algebra, would yield an approximation of any $h \in L^+$ by $I(\varphi)^{-1}(h * \varphi)$, where φ is "sufficiently concentrated" around e. In the next step one should write

$$h = \sum_{i=1}^{n} h_i ,$$

where each h_i is "sufficiently concentrated" around some point s_i (decomposition of unity). By application of the theorem on (left) approximate identities once more, one would obtain an approximation of $h*\varphi$ by

$$\sum_{i=1}^{n} I(h_i)\varphi_{s_i} .$$

The obvious defect of this procedure, is its dependence on the Haar integral. This defect, however, is not so severe as it may appear at first sight. In fact one may apply an approximate integral $f \to (\overline{f:\psi})$, and procede as outlined above. The details follow.

Henceforth we shall use the simplified notation $(f:\varphi)$ to mean $(\overline{f:\varphi})$. This will not cause any confusion, since no lower estimates $(\underline{f:\varphi})$ will appear in the sequel.

For every $\varphi \in L^+$, $\varphi \neq 0$, we define the *convolution relatively to* φ on L^+, by

$$[f*g]_\varphi(x) = \big(f(s)g(s^{-1}x):\varphi(s)\big) = \big(f(xs)g(s^{-1}):\varphi(s)\big) . \tag{3.1}$$

PROPOSITION 3.1. *For fixed, non-zero $f, g \in L^+$, the functions*

$$h_\varphi(x) = \frac{[f*g]_\varphi(x)}{(f:\varphi)}, \qquad \varphi \in L^+,\ \varphi \neq 0 . \tag{3.2}$$

are equicontinuous.

PROOF. Let $x \in G$ and $\varepsilon > 0$ be arbitrary. The functions $s \to g(s^{-1}y)$ converge uniformly to $s \to g(s^{-1}x)$ when y tends to x, and they will vanish off some fixed compact set for all y in a compact neighbourhood of x. By Proposition 1.2, there is a $V \in \mathscr{V}$ such that for every $y \in xV$, and every $\varphi \in L^+$, $\varphi \neq 0$,

$$\left| 1 - \frac{[f*g]_\varphi(y)}{[f*g]_\varphi(x)} \right| < \frac{\varepsilon}{[f*g]_f(x)} . \tag{3.3}$$

By (2.2), one has

$$\big(f(s)g(s^{-1}x):\varphi(s)\big) \leq \big(f(s)g(s^{-1}x):f(s)\big)\big(f(s):\varphi(s)\big) . \tag{3.4}$$

Multiplying both sides of (3.3) by

$$\frac{[f*g]_\varphi(x)}{(f:\varphi)} ,$$

and making use of (3.4), one obtains

$$\left| \frac{[f*g]_\varphi(x)}{(f:\varphi)} - \frac{[f*g]_\varphi(y)}{(f:\varphi)} \right| < \varepsilon$$

for all $y \in xV$ and all $\varphi \in L^+$, $\varphi \neq 0$.

By (1.7) and (1.8), the relative convolution is *positively homogeneous* and *sub-additive.* The latter statement means that

$$(3.5) \qquad \left[\sum_{i=1}^{n} f_i * g\right]_\varphi \leqq \sum_{i=1}^{n} [f_i * g]_\varphi$$

for all $f,g,\varphi \in L^+$ and $\varphi \neq 0$.

When φ is concentrated around e, the relative convolution is *approximately additive* in the sense of

PROPOSITION 3.2. *Let $f_1,\ldots,f_n$, and g be arbitrary non-zero members of L^+, and write $f = \sum_{i=1}^n f_i$. For every $\varepsilon > 0$, there is a $V \in \mathscr{V}$ such that*

$$(3.6) \qquad \sum_{i=1}^{n} [f_i * g]_\varphi(x) - [f*g]_\varphi(x) < \varepsilon(f:\varphi)\,,$$

for all $\varphi \in L_V^+$ and all $x \in G$.

PROOF. We define $k_\varphi(x)$ by

$$k_\varphi(x) = \frac{\sum_{i=1}^{n} [f_i * g]_\varphi(x)}{(f:\varphi)} - \frac{[f*g]_\varphi(x)}{(f:\varphi)}\,,$$

and we shall prove that there exists a $V \in \mathscr{V}$ such that $k_\varphi(x) < \varepsilon$ for all $\varphi \in L_V^+$, $\varphi \neq 0$, and all $x \in G$.

By Proposition 1.1, there exists for every $x \in G$ a $V_x \in \mathscr{V}$ such that $k_\varphi(x) < \frac{1}{2}\varepsilon$ for all $\varphi \in L_{V_x}^+$. By Proposition 3.1, the functions k_φ are equicontinuous. Hence there exists an open neighbourhood U_x of each point $x \in G$, such that $k_\varphi(y) < \varepsilon$ for all $y \in U_x$ and all $\varphi \in L_{V_x}^+$. The functions k_φ all vanish outside the compact set $K = \mathrm{spt}(f)\cdot\mathrm{spt}(g)$. Let $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$, and define $V = V_{x_1} \cap \ldots \cap V_{x_n}$. For every $\varphi \in L_V^+$, $\varphi \neq 0$, and every $x \in G$, one has $k_\varphi(x) < \varepsilon$. This proves the proposition.

We now turn to the key lemma, on the existence of approximate identities under relative convolutions.

PROPOSITION 3.3. *Let $g \in L^+$ and $\varepsilon > 0$ be arbitrary. Then there exists a $U \in \mathscr{V}$ such that*

(3.7) $$\|[h*g]_\varphi - (h:\varphi)g_t\|_\infty \leqq (h:\varphi)\varepsilon ,$$

whenever $t \in G$, $h \in L^+_{tU}$, $\varphi \in L^+$, $\varphi \neq 0$.

Similary there exists a $V \in \mathscr{V}$ *such that*

(3.8) $$\|[g*k]_\varphi - (k:\varphi)g^u\|_\infty \leqq (k^*:\varphi)\varepsilon ,$$

whenever $u \in G$, $k \in L^+_{Vu^{-1}}$, $\varphi \in L^+$, $\varphi \neq 0$.

PROOF. 1) By right uniform continuity, there is a $U \in \mathscr{V}$ such that

$$xy^{-1} \in U \quad \Rightarrow \quad |g(x)-g(y)| < \varepsilon .$$

Let $t \in G$ and $h \in L^+_{tU}$. Now, $h(s)=0$ if $s \notin tU$, or equivalently if $(t^{-1}x)(s^{-1}x)^{-1} = t^{-1}s \notin U$, where $x \in G$ is arbitrary. Hence for every pair $s,x \in G$:

$$|h(s)g(s^{-1}x) - h(s)g(t^{-1}x)| \leqq \varepsilon h(s) .$$

Writing this as two inequalities and applying (1.8) to both, one obtains the desired relation (3.7).

2) By left uniform continuity, there is a $V \in \mathscr{V}$ such that

$$x^{-1}y \in V \quad \Rightarrow \quad |g(x)-g(y)| < \varepsilon .$$

Let $u \in G$ and $k \in L^+_{Vu^{-1}}$. Now, $k^*(s)=0$ if $s^{-1} \notin Vu^{-1}$, or equivalently if $(xs)^{-1}(xs) = s^{-1}u \notin V$, where $x \in G$ is arbitrary. Hence for every pair $s,x \in G$:

$$|g(xs)k^*(s) - g(xu)k^*(s)| \leqq \varepsilon k^*(s) .$$

Again, the desired relation (3.8) follows by application of (1.8).

PROPOSITION 3.4. *Let* $f,g \in L^+$ *and* $\varepsilon > 0$ *be arbitrary. Then there exists a* $V \in \mathscr{V}$ *such that every* $\varphi \in L^+_V$ *admits group elements* $t_1,\ldots,t_n \in \operatorname{spt}(f)$ *and positive numbers* $\alpha_1,\ldots,\alpha_n$ *such that*

(3.9) $$\left\| [f*g] - \sum_{i=1}^n \alpha_i g_{t_i} \right\|_\infty \leqq (f:\varphi)\varepsilon .$$

PROOF. By Proposition 3.3, there is a $U \in \mathscr{V}$ such that

(3.10) $$\|[h*g]_\varphi - (h:\varphi)g_t\|_\infty \leqq (h:\varphi)\tfrac{1}{4}\varepsilon$$

whenever $t \in G$, $h \in L^+_{tU}$, $\varphi \in L^+$, $\varphi \neq 0$. Let

$$\operatorname{spt}(f) \subset \bigcup_{i=1}^n t_iU, \quad \text{where} \quad t_1,\ldots,t_n \in \operatorname{spt}(f) .$$

By decomposition of unity, one may write $f=\sum_{n}^{i=1} f_i$, where $f_i \in L^+_{t_iU}$ for $i=1,\ldots,n$. By Proposition 1.1 and Proposition 3.2, there is a $V \in \mathscr{V}$ such that

$$\sum_{i=1}^{n} (f_i:\varphi) \leqq 2(f:\varphi)\,, \tag{3.11}$$

and

$$\left\| [f*g]_\varphi - \sum_{i=1}^{n} [f_i*g]_\varphi \right\|_\infty \leqq (f:\varphi)\tfrac{1}{2}\varepsilon \tag{3.12}$$

for all $\varphi \in L^+_V$, $\varphi \neq 0$. Applying (3.10) with $h=f_i$, $i=1,\ldots,n$, and adding the resulting inequalities with use of triangle inequality and (3.11), one obtains

$$\left\| \sum_{i=1}^{n} [f_i*g]_\varphi - \sum_{i=1}^{n} (f_i:\varphi) g_{t_i} \right\|_\infty \leqq (f:\varphi)\tfrac{1}{2}\varepsilon$$

for all $\varphi \in L_V{}^+$, $\varphi \neq 0$. Combination of this result and (3.12) yields the desired relation (3.9) with $\alpha_i=(f_i:\varphi)$ for $i=1,\ldots,n$.

PROPOSITION 3.5 (Cartan). *For every non-zero $f \in L^+$ and every $\varepsilon>0$, there is a $V \in \mathscr{V}$ such that every $g \in L^+_V$ admits group elements $t_1,\ldots,t_n \in \operatorname{spt}(f)$ and positive numbers $\gamma_1,\ldots,\gamma_n$ such that*

$$\left\| f - \sum_{i=1}^{n} \gamma_i g_{t_i} \right\|_\infty \leqq \varepsilon\,. \tag{3.13}$$

PROOF. By Proposition 3.3, there is a $V \in \mathscr{V}$ such that the following relation holds for an arbitrary, but henceforth fixed, $g \in L^+_V$, and for every $\varphi \in L^+$, $\varphi \neq 0$:

$$\|[f*g]_\varphi - (g^*:\varphi)f\|_\infty \leqq (g^*:\varphi)\tfrac{1}{2}\varepsilon\,. \tag{3.14}$$

By Proposition 3.4, one may find a $\varphi \in L^+$, group elements $t_1,\ldots,t_n \in \operatorname{spt}(f)$, and positive numbers $\alpha_1,\ldots,\alpha_n$ such that

$$\left\| [f*g]_\varphi - \sum_{i=1}^{n} \alpha_i g_{t_i} \right\|_\infty \leqq \frac{(f:\varphi)}{(f:g^*)}\frac{\varepsilon}{2}. \tag{3.15}$$

Combination of (3.14) and (3.15) gives

$$\left\| (g^*:\varphi)f - \sum_{i=1}^{n} \alpha_i g_{t_i} \right\|_\infty \leqq \left((g^*:\varphi) + \frac{(f:\varphi)}{(f:g^*)} \right) \frac{\varepsilon}{2}.$$

Dividing through by $(g^*:\varphi)$, writing $\gamma_i=(g^*:\varphi)^{-1}\alpha_i$ for $i=1,\ldots,n$, and making use of the relation $(f:\varphi) \leqq (f:g^*)(g^*:\varphi)$, one obtains the desired relation (3.13).

PROPOSITION 3.6. *The separation property* (S) *is valid in the locally compact group* G.

PROOF. Let $f,g \in L^+$ and let $f(x) < g(x)$ for $x \in \mathrm{spt}(f)$. Define $h = \frac{1}{2}(f+g)$, and

$$\varepsilon = \tfrac{1}{2} \inf \{g(x) - h(x) \mid x \in \mathrm{spt}(h)\} .$$

The set $\{x \mid g(x) > \varepsilon\}$ is open and contains $\mathrm{spt}(h)$. By a known property of topological groups (or actually of uniform spaces), there is a $U \in \mathscr{V}$ such that $\mathrm{spt}(h) \cdot U \subset \{x \mid g(x) > \varepsilon\}$. Writing $A = \mathrm{spt}(h) \cdot U$, we can state

$$\text{(3.16)} \qquad k \in L_A^+, \ \|h-k\|_\infty \leqq \varepsilon \quad \Rightarrow \quad f \leqq k \leqq g .$$

By Proposition 3.5 there is a $V \in \mathscr{V}$, $V \subset U$, such that every $\varphi \in L_V^+$ admits group elements $t_1, \ldots, t_n \in \mathrm{spt}(f)$ and positive numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{i=1}^n \alpha_i g_{t_i} \in L_A^+, \qquad \left\| h - \sum_{i=1}^n \alpha_i g_{t_i} \right\|_\infty \leqq \varepsilon .$$

By (3.16) the proof is completed.

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