NOTE ON RAMANUJAN’S FUNCTION \( \tau(n) \)

O. KOLBERG

The function \( \tau(n) \) is defined by

\[
\sum_{1}^{\infty} \tau(n)x^n = x \prod_{1}^{\infty} (1-x^n)^{24}.
\]

It is well known that for certain moduli the residue of \( \tau(n) \) can be expressed by the function \( \sigma_k(n) \), the sum of the \( k \)-th powers of the divisors of \( n \). The strongest results obtained in this direction are the following:

(1) \( \tau(8n+1) \equiv \sigma_{11}(8n+1) \pmod{2^{11}} \),
(2) \( \tau(8n+3) \equiv 1217\sigma_{11}(8n+3) \pmod{2^{13}} \),
(3) \( \tau(8n+5) \equiv 1537\sigma_{11}(8n+5) \pmod{2^{12}} \),
(4) \( \tau(8n+7) \equiv 705\sigma_{11}(8n+7) \pmod{2^{14}} \),
(5) \( \tau(3n+1) \equiv \sigma_{11}(3n+1) \pmod{3^6} \),
(6) \( \tau(3n+2) \equiv 53\sigma_{11}(3n+2) \pmod{3^6} \),
(7) \( \tau(n) \equiv 5n^2\sigma_3(n) - 4n\sigma_9(n) \pmod{5^3} \) if \( (n,5) = 1 \),
(8) \( \tau(n) \equiv n\sigma_3(n) \pmod{7} \),
(9) \( \tau(n) \equiv 0 \pmod{23} \) if \( (n/23) = -1 \),
(10) \( \tau(n) \equiv \sigma_{11}(n) \pmod{691} \),

where \( (n/p) \) is Legendre’s symbol. The congruences (1)–(6) were proved in [3], (7) is due to Bambah and Chowla [1], (8) to Wilton [5], (9) to Hardy [2], and (10) to Ramanujan [4].

The object of this note is to prove the congruence

(11) \( \tau(n) \equiv n\sigma_9(n) \pmod{7^2} \) if \( (n/7) = -1 \).

We put

\[
P = 1 - 24 \sum \sigma(n)x^n, \quad Q = 1 + 240 \sum \sigma_3(n)x^n, \quad R = 1 - 504 \sum \sigma_5(n)x^n,
\]

where \( \sigma(n) = \sigma_1(n) \), and the sums are taken from 1 to \( \infty \). It is known (cf. [4]) that

\[
1 + 480 \sum \sigma_7(n)x^n = Q^2,
1008 \sum n\sigma_5(n)x^n = Q^2 - PR,
1584 \sum n\sigma_9(n)x^n = 3Q^3 + 2R^2 - 5PQR,
1728 \sum \tau(n)x^n = Q^3 - R^2.
\]

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Combining these equations, and noticing that \( R \equiv 1 \pmod{7} \), we easily verify the congruence

\[ 7 + \sum \{ n\sigma_9(n) + 13\tau(n) \} x^n \equiv 7Q(1 + 4 \sum \{ \sigma_7(n) - n\sigma_6(n) \} x^n) \pmod{7^2}. \]

Now, if \((n, 7) = 1\) we have

\[ n\sigma_9(n) = n^6\sigma_{-5}(n) \equiv \sigma(n) \pmod{7}. \]

Since \( \sigma_7(n) \equiv \sigma(n) \pmod{7} \) for all \( n \), we get

\[ \sigma_7(n) - n\sigma_6(n) \equiv \begin{cases} 0 & \pmod{7} \quad \text{if } (n, 7) = 1 \\ \sigma(n) & \pmod{7} \quad \text{if } (n, 7) = 7. \end{cases} \]

Thus, returning to (12) we obtain

\[ 7 + \sum \{ n\sigma_9(n) + 13\tau(n) \} x^n \equiv 7(1 + 2 \sum \sigma_3(n)x^n)(1 + 4 \sum \sigma(7n)x^{7n}) \pmod{7^2}. \]

Further, if \((n, 7) = 1\), we have

\[ \sigma_3(n) = n^3\sigma_{-3}(n) \equiv n^3\sigma_3(n) \pmod{7}, \]

and hence

\[ \sigma_3(n) \equiv 0 \pmod{7} \quad \text{if } (n/7) = -1. \]

From (13) and (14) we conclude that

\[ n\sigma_9(n) + 13\tau(n) \equiv 0 \pmod{7^2} \quad \text{if } (n/7) = -1, \]

which implies (11), cf. (8) and (14).

**Added in Proof:** In connection with formula (8) it may be noticed that D. H. Lehmer has shown that

\[ (n^3 - 1) \tau(n) \equiv 30n\sigma_{16}(n) + 16n\sigma_9(n) - (12n^4 - 15n) \sigma_3(n) \pmod{7^2}. \]

**References**